

On some applications of the essential norms of singular integral operators

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Abstract. In this paper, we calculate the norms and essential norms of singular operators with the Cauchy kernel. We also determine the conditions under which these quantities coincide. Using the essential norms, conditions are obtained under which singular integral operators with bounded coefficients are Noetherian in the weighted space L_2 . In the absence of a weights, the presented conditions coincide with the corresponding conditions of I. Simonenko.

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Asupra unor aplicații ale normelor esențiale ale operatorilor integrali singulari

Rezumat. În această lucrare se calculează normele și normele esențiale ale operatorilor singulari cu nucleu Cauchy. Se determină condițiile în care aceste mărimi coincid. Folosind normele esențiale, se obțin condiții în care operatorii integrali singulari cu coeficienți mărginiți sunt noetherieni în spațiul L_2 cu ponderi. În absența ponderii, aceste condiții coincid cu condițiile respective ale lui I. Simonenko.

Cuvinte-cheie: operatori integrali singulari, operatori noetherieni, contur de tip Lyapunov, norme esențiale.

1. INTRODUCTION

Let Γ be a Lyapunov contour on the \mathbb{C} plane, and S_Γ be the operator of singular integration along Γ :

$$(S_\Gamma \varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau \quad (t \in \Gamma).$$

Here, the integral is understood in the sense of the principal value. We agree on the following notation:

$$\|\varphi\|_{p,\rho} = \left(\int_\Gamma |\varphi(t)|^p \rho(t) |dt| \right)^{1/p} \quad (1)$$

the norm in the space $L_p(\Gamma, \rho)$; $\|\varphi\|_p = \|\varphi\|_{p-1}$; if $A \in L(E)$, then

$$\|A\| = \inf \|A + T\| \quad (T \in \mathfrak{T}(E)), \quad (2)$$

the indices p, ρ of the norms $\|A\|$ and $|A|$ will emphasize the space $L_p(\Gamma, \rho)$ in which the operator A acts as follows

$$\gamma(p) = \begin{cases} ctg \frac{\pi}{2p}, & \text{if } 1 < p \leq 2, \\ tg \frac{\pi}{2p}, & \text{if } 2 \leq p < \infty. \end{cases} \quad (3)$$

M. Riesz [1] proved the boundedness of the operator S_{Γ_0} in the space $L_p(\Gamma_0)$, where Γ_0 is the unit circle and $1 < p < \infty$. Then, Hardy and Littlewood [2], and Babenko [3] transferred this result to the space $L_p(R, |x|^\beta)$ ($1 < p < \infty$, $0 < \beta < 1$). In the paper [4], the boundedness of the operator S_Γ in the space $L_p(\Gamma, \rho)$ was proved for the case when Γ is a Lyapunov contour and

$$\rho(t) = \prod_{k=1}^n |t - t_k|^{\beta_k} \quad (t_k \in \Gamma, -1 < \beta_k < p - 1).$$

The condition $-1 < \beta_k < p - 1$ is also necessary for the boundedness of the operator S_Γ in $L_p(\Gamma, \rho)$. This is confirmed by

Lemma 1.1. *Let $S_\Gamma \in L(L_p(\Gamma, \rho))$, then $\rho \in l(L_p(\Gamma))$ and $\rho^{-1} \in L_q(\Gamma)$ $\left(q = \frac{p}{p-1}\right)$.*

Proof. The boundedness of the operator S_Γ in the space $L_p(\Gamma, \rho)$ implies the boundedness in $L_p(\Gamma)$ of the operator $M = \pi i \rho(S_\Gamma R - R S_\Gamma) \rho^{-1}$, where $(R\varphi)(t) = t\varphi(t)$. But

$$(M\varphi)(t) = \rho(t) \int_\Gamma \rho^{-1}(t)\varphi(t) dt,$$

therefore $\rho \in L_p(\Gamma)$, $\rho^{-1} \in L_q(\Gamma)$. □

It was later discovered that the norms and essential norms of singular operators in weighted L_p spaces are of great theoretical and applied interest. In particular, the problem of using these quantities to find criteria for the Noetherian property of singular integral operators with bounded coefficients should be noted. This problem has been investigated by many mathematicians, especially after the appearance of Simonenko's works [5], [6]. In these papers, in particular, the norm of the operator S_Γ is used to obtain conditions for the Noetherian property of a singular operator in L_2 . Naturally, the problem of generalizing these results to weighted L_2 spaces arose. In the present paper, we prove a theorem that allows us to express the conditions for the Noetherian property of a characteristic singular operator in a weighted space through the conditions for the

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Noetherian property of another singular operator in an unweighted space. Using this theorem and the results of Simonenko's work [6], conditions for the Noetherian property of singular operators in the space L_2 with weight are obtained. These conditions are not only sufficient but also necessary for the singular operator to be Noetherian. The present work is devoted to the presentation of these results.

2. RELATIONSHIP BETWEEN THE NORM AND THE ESSENTIAL NORM
OF THE OPERATOR S_Γ

Let Γ consist of a finite number of simple closed disjoint Lyapunov curves, $t_1, t_2, \dots, t_n \in \Gamma$ and

$$\rho(t) = \prod_{k=1}^n |t - t_k|^{\beta_k}, \quad p > 1, \quad -1 < \beta_k < p - 1, \quad r_0 = p, \quad r_k = p(1 + \beta_k)^{-1},$$

$$(k = 1, 2, \dots, n), \quad \bar{r}_k = \max(r_k, r_k(r_k - 1)^{-1}), \quad r = \max(\bar{r}_0, \bar{r}_1, \dots, \bar{r}_n).$$

In the works [7], [8], it is shown that

$$|S_\Gamma|_{p,\rho} \geq ctg \frac{\pi}{2r}. \quad (4)$$

We present one sufficient condition under which the equality $\|A\| = |A|$ holds (see [7], [8], [9]).

Theorem 2.1. *Let for some operator $A \in L(E)$ there exists a sequence of operators $B_n \in L(E)$ such that*

- (1) $\|B_n \varphi\| = \|\varphi\|$, $\varphi \in E$;
- (2) $AB_n = B_n A$, $\forall n \in \mathbb{N}$;
- (3) B_n weakly converges to zero.

Then $\|A\| \geq |A|$.

Proof. It suffices to prove that, for every compact operator T , the inequality

$$\|A + T\| \geq \|A\| \quad (5)$$

holds.

Let $\varepsilon > 0$ be arbitrary and φ be some vector in E , satisfying the conditions $\|\varphi\| = 1$, $\|A\varphi\| > \|A\| - \varepsilon$. We set $\varphi_n = B_n \varphi$. Since $B_n \rightarrow 0$ weakly, we have $\|TB_n \varphi\| \rightarrow 0$. We choose n such that $\|TB_n \varphi\| < \varepsilon$, then

$$\|(A + T)\varphi_n\| \geq \|AB_n \varphi\| - \varepsilon = \|A\varphi\| - \varepsilon \geq (\|A\| - 2\varepsilon)\|\varphi_n\|,$$

hence $\|A + T\| \geq \|A\|$. The theorem is proved. □

Denote the unit circle by Γ_0 , $\Gamma_0 = \{t : |t| = 1\}$.

Theorem 2.2. *For each $p \in (1, \infty)$, the equality*

$$|S_0|_p = \|S_0\|_p = \gamma(p) \quad (S_0 = S_{\Gamma_0}). \quad (6)$$

Proof. In [10], the norm of the operator was calculated

$$(H\varphi)(t) = \frac{1}{2\pi} \int_0^{2\pi} ctg \frac{\tau - t}{2} \varphi(\tau) d\tau,$$

in the real space $L_p(0, 2\pi)$. It turned out that $\|H\|_p = \gamma(p)$. After some time, in [11], it was proved that this equality is also valid in the complex space $L_p(0, 2\pi)$. Let α be an isometry $L_p(\Gamma_0) \rightarrow L_p(0, 2\pi)$, defined by the equality $(\sigma\varphi)(\theta) = \varphi(\exp(i\theta))$. It is directly verified that $\sigma H \sigma^{-1} = i(K_0 - S_0)$, where K_0 is a one-dimensional operator,

$$(K_0\varphi)(t) = \frac{1}{2\pi} \int_{\Gamma_0} \varphi(\tau) |dt|.$$

It follows that $\|S_0 - K_0\|_p = \gamma(p)$ and $|S_0|_p \leq \gamma(p)$. Taking into account inequality (4), we obtain the equality $|S_0|_p = \gamma(p)$. Consider the sequence of operators $(B_n\varphi)(t) = t^n\varphi(t^n)$. It is directly verified that the sequence (B_n) and the operator S_0 satisfy the conditions of Theorem 2.1 in the space $L_p(\Gamma_0)$ and, consequently, $|S_0|_p = \|S_0\|_p$. The theorem is proved. \square

Remark 2.1. *Let Γ be a simple closed Lyapunov contour. If $|S_\Gamma|_2 = \|S_\Gamma\|_2$, then Γ is a circle.*

Indeed, let Γ not be a circle, then in [11] $S_\Gamma^* \neq S_\Gamma$. Since $S_\Gamma^2 = I$, then $S_\Gamma S_\Gamma^* \neq I$, hence the spectrum of the operator $S_\Gamma S_\Gamma^*$ contains a point $\lambda \neq 1$. From the equality $S_\Gamma S_\Gamma^* - \lambda I = S_\Gamma^*(I - S_\Gamma^* S_\Gamma \lambda) S_\Gamma$ it follows that the spectrum of the operator $S_\Gamma S_\Gamma^*$ contains a point $\lambda_0 > 1$. Hence, $|S_\Gamma|_2 > 1$. On the other hand, (see [11]), $|S_\Gamma|_2 = |S_0|_2 = 1$. The resulting contradiction proves that Γ is a circle.

Let us consider two more examples of calculating the norms of the operator S_Γ .

Example 2.1. Let $\Gamma = \mathbb{R}$, $\rho(x) = |x|^\beta$ ($-1 < \beta < p - 1$); $A = aI + bS_{\mathbb{R}}$, ($a, b \in \mathbb{C}$) and

$$(B_n\varphi)(x) = n^{\frac{1+\beta}{p}} \varphi(nx).$$

It is directly verified that the operators A and B_n satisfy the conditions of Theorem 2.1, therefore,

$$\|aI + bS_{\mathbb{R}}\|_{p,\rho} = |aI + bS_{\mathbb{R}}|_{p,\rho}.$$

In particular,

$$\|S_{\mathbb{R}}\|_{p,|x|^\beta} = |S_{\mathbb{R}}|_{p,|x|^\beta}. \quad (7)$$

Similar equalities can be written for projectors

$$P_{\mathbb{R}} = \frac{1}{2}(I + S_{\mathbb{R}}), \quad Q_{\mathbb{R}} = \frac{1}{2}(I - S_{\mathbb{R}}).$$

Example 2.2. Let us show that

$$\|aI + bS_0\|_{p, \hat{\rho}} = |aI + bS_0|_{p, \hat{\rho}}, \quad (8)$$

where $a, b \in \mathbb{C}$ and $\hat{\rho}(t) = |t - 1|^{p-2-\beta} \cdot |t + 1|^{\beta}$.

Consider the operator B , defined by the equality

$$(B\varphi)(t) = \frac{2^{1/p}}{t-1} \varphi\left(i \frac{t+1}{t-1}\right)$$

and let us prove that it isometrically maps the space $L_p(\mathbb{R}; |x|^{\beta})$ onto $L_p(\Gamma_0; |t-1|^{p-2-\beta} \cdot |t+1|^{\beta})$. Furthermore, let us also prove that the equality $BS_{\mathbb{R}}B^{-1} = -S_0$ holds. Indeed, let $\varphi \in L_p(\mathbb{R}; |x|^{\beta})$, then

$$(BS_{\mathbb{R}}\varphi)(t) = \frac{1}{\pi i} \left(B \int_{-\infty}^{+\infty} \frac{\varphi(x)}{x-y} dx \right) (t) = \frac{2^{1/p}}{\pi i} \frac{1}{t-1} \int_{-\infty}^{+\infty} \frac{\varphi(x)}{x - i \frac{t+1}{t-1}} dx.$$

In the last integral we will make a change of variables

$$x = i \frac{\tau + 1}{\tau - 1}, \quad dx = -i \frac{2}{(\tau - 1)^2} d\tau, \quad t \in \Gamma_0,$$

then

$$(BS_{\mathbb{R}}\varphi)(t) = -\frac{2^{1/p}}{\pi i} \frac{1}{t-1} \int_{\Gamma_0} \frac{\varphi\left(i \frac{\tau+1}{\tau-1}\right)}{\frac{\tau+1}{\tau-1} - \frac{t+1}{t-1}} \cdot \frac{2}{(\tau-1)^2} d\tau = -\frac{1}{\pi i} \int_{\Gamma_0} \frac{(B\varphi)(\tau)}{\tau-t} d\tau.$$

Thus, we obtained the equality $BS_{\mathbb{R}}\varphi = -S_0B\varphi$, i.e., $BS_{\mathbb{R}}B^{-1} = -S_0$. Let us prove the isometry. We have

$$\begin{aligned} \|\varphi\|_{L_p(\mathbb{R}; |x|^{\beta})}^p &= \int_{-\infty}^{+\infty} |\varphi(x)|^p |x|^{\beta} dx = \int_{\Gamma_0} \left| \varphi\left(i \frac{t+1}{t-1}\right) \right|^p \left| \frac{t+1}{t-1} \right|^{\beta} \left| \frac{2}{(t-1)^2} \right| dt \\ &= \int_{\Gamma_0} |(B\varphi)(t)|^p |t-1|^{p-2-\beta} \cdot |t+1|^{\beta} dt = \|B\varphi\|_{L_p(\Gamma_0; |t-1|^{p-2-\beta} \cdot |t+1|^{\beta})}^p. \end{aligned}$$

From the obtained properties of the operator B and from equality (7) it follows

$$\|aI + bS_0\|_{p, \hat{\rho}} = |aI + bS_0|_{p, \hat{\rho}}, \quad \text{where } \hat{\rho}(t) = |t - 1|^{p-2-\beta} \cdot |t + 1|^{\beta}. \quad (8)$$

3. NOETHERIAN CONDITIONS FOR THE OPERATOR $aP + Q$ IN THE SPACE $L_2(\Gamma)$

Let Γ be a closed contour of Lyapunov type and A be the operator defined by the equality

$$A = aP + Q, \quad (9)$$

where a is a measurable and bounded function on the contour Γ ,

$$P = \frac{1}{2}(I + S), \quad Q = I - P$$

and S is the operator

$$(S\varphi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau \quad (t \in \Gamma).$$

In this Section we will use Simonenko's local principle and the essential norms of the operator S to determine the Noetherian conditions for the operator (9). First, we will reduce the study of the operator A to the study of a simpler operator. We consider the condition $\text{ess inf } |a(t)| > 0$ to be fulfilled, otherwise the operator A is not (see [11]) Noetherian. We express the function a in the form

$$a(t) = |a(t)| \cdot e^{i\varphi(t)}, \quad \varphi(t) = \arg a(t).$$

As the function $a(t)$ ($\text{ess inf } |a(t)| > 0$) admits a factorization (see [11])

$$|a(t)| = a_+(t) \cdot a_-(t), \quad a_+(t) = \exp(P \ln |a(t)|),$$

$$a_-(t) = \exp(Q \ln |a(t)|), \quad a_+^{\pm 1} \in L_{\infty}^+(\Gamma), \quad a_-^{\pm 1} \in L_{\infty}^-(\Gamma),$$

the operator $A = aP + Q$ can be written in the form

$$A = a_-(e^{i\varphi}P + Q)(a_+P + a_-^{-1}Q), \quad (10)$$

where the operators a_-I and $a_+P + a_-^{-1}Q$ are invertible in $L_p(\Gamma)$:

$$(a_-I)^{-1} = a_-^{-1}I, \quad (a_+P + a_-^{-1}Q)^{-1} = a_+^{-1}P + a_-Q.$$

Therefore, from equality (10), the following theorem follows.

Theorem 3.1. *The operator $A = aP + Q$ is Noetherian in $L_p(\Gamma)$ if and only if the operator $e^{i\varphi}P + Q$ is Noetherian in this space. Moreover,*

$$\text{Ind}(aP + Q) = \text{Ind}(e^{i\varphi}P + Q).$$

Thus, Theorem 3.1 reduces the study of operators of the form $aP + Q$ to operators of the form $e^{i\varphi(t)}P + Q$, where $\varphi(t)$ is real, bounded, and measurable. We also mention an important property of the operators $A = aP + Q$, $a(t) \geq \delta > 0$, $\forall t \in \Gamma$.

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Theorem 3.2. *The operator $A = bP + Q$, $b \in L_\infty(\Gamma)$ is Noetherian in $L_p(\Gamma)$ if and only if the operator $C = abP + Q$, $a(t) \geq \delta > 0$, $t \in \Gamma$ is Noetherian and $\text{Ind } B = \text{Ind } C$.*

The statement of the theorem follows immediately from the fact that the function a admits a factorization, $a = a_+ \cdot a_-$ and from the equality

$$C = a_-(bP + Q)(a_+P + a_-^{-1}Q).$$

We denote by $\mathfrak{S}_2(\Gamma)$ the set of all functions $a \in L_\infty(\Gamma)$ that verify the condition: for any $\tau \in \Gamma$ there is a neighborhood $u(\tau) \subset \Gamma$ and two functions $g_\tau^\pm \in L_\infty^\pm(\Gamma)$, such that the set of values of the function $g_\tau^-(t)a(t)g_\tau^+(t)$, for $t \in u(\tau)$, lies in a closed half-plane Λ_τ that does not contain the origin of coordinates (Figure 1).

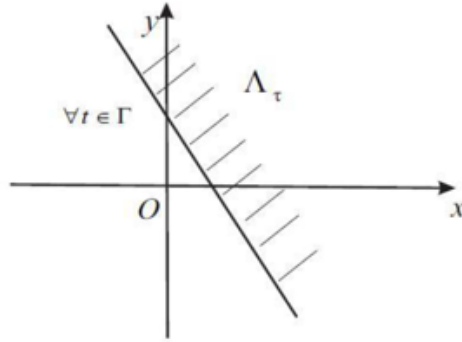


Figure 1. The set Λ_τ

The simplest function that satisfies these conditions is the function $a(t) = e^{i\varphi(t)}$, where $\varphi(t)$ is a real function and $\sup |\varphi(t)| < \frac{\pi}{2}$. The above condition is satisfied for this function with $g_\tau^+(t) \equiv 1$. The set of values of this function lies in the right half-plane, $\{z \mid \text{Re } z > 0\}$.

Theorem 3.3. *If $a \in \mathfrak{S}_2(\Gamma)$, then the operator $A = aP + Q$ is Noetherian in the space $L_2(\Gamma)$.*

Proof. Let t be an arbitrary point on Γ , $g_\tau^\pm \in L_\infty^\pm(\Gamma)$ and $u(\tau)$ be the neighborhood of the point τ such that for $t \in u(\tau)$ the set of values of the function a lies in the half-plane Λ_τ . We denote by $a_\tau(t)$ the function

$$a_\tau(t) = \begin{cases} a(t), & \text{for } t \in u(\tau), \\ \alpha [g_\tau^-(t)g_\tau^+(t)]^{-1}, & \text{for } t \in \Gamma \setminus u(\tau), \end{cases}$$

where α is some complex number in Λ_τ . We will show that the operator $a_\tau P + Q$ is Noetherian in $L_2(\Gamma)$. Let $b(t) = g_\tau^-(t)a_\tau(t)g_\tau^+(t)$. The set of values of the function b for any $t \in \Gamma$ lies in Λ_τ and is bounded:

$$b(t) = \begin{cases} g_\tau^-(t)a(t)g_\tau^+(t), & \text{for } t \in u(\tau), \\ \alpha, & \text{for } t \in \Gamma \setminus u(\tau). \end{cases}$$

Then there is a number $\gamma \in \mathbb{C}$ such that

$$\sup_{t \in \Gamma} |\gamma b(t) - 1| < 1.$$

We write the operator $A_\tau = a_\tau P + Q$ in the form

$$A_\tau = (g_\tau^-)^{-1}[(\gamma b - 1)P + I][(\gamma g_\tau^+)^{-1}P + g_\tau^- Q]. \quad (11)$$

In equality (11) the operators $(g_\tau^-)^{-1}I$ and $(\gamma g_\tau^+)^{-1}P + g_\tau^- Q$ are invertible, their inverses have the form $(g_\tau^-)I$ and, respectively, $\gamma g_\tau^+ P + (g_\tau^-)^{-1}Q$. From Theorem 2.2 we deduce that the essential norm of the operator P is equal to one, $|P|_{L_2(\Gamma)} = 1$. Then

$$|(\gamma b - 1)P|_{L_2(\Gamma)} = \inf_{T \in \mathcal{K}} \|(\gamma b - 1)P + T\|_{L_2(\Gamma)} \leq \sup_{t \in \Gamma} |\gamma b(t) - 1| < 1. \quad (12)$$

Thus, the operator $(\gamma b - 1)P + I$ is Noetherian, and consequently the operator A_τ , defined by equality (11), is Noetherian. At any point $\tau \in \Gamma$, the function a is equivalent to the function a_τ and the operator $a_\tau P + Q$ is Noetherian in $L_2(\Gamma)$. Based on the results of the work of Simonenko [6], we deduce that $A = aP + Q$ is also Noetherian in the space $L_2(\Gamma)$. Theorem 3.3 is proved. \square

Remark 3.1. *We mention that in the proof of Theorem 3.3, an important role was played by the fact that we know the size of the essential norm of the operator P : $|P|_{L_2(\Gamma)} = 1$. This observation is also valid in what follows.*

4. NOETHERIAN CRITERIA FOR THE OPERATOR $aP + Q$ IN THE SPACE L_p WITH WEIGHTS

To establish other Noetherian criteria for singular integral operators in the spaces L_p and L_p with weights, we recall some important notions from the theory of operators of the form $aP + bQ$ with measurable and bounded coefficients, notions that can be found in the works of I. Simonenko, I. Gohberg and N. Krupnik (see for example [5], [11]).

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Definition 4.1. A function $a \in L_\infty(\Gamma)$ is said to admit a factorization in the space $L_p(\Gamma, \rho)$ if it can be expressed in the form

$$a(t) = a_+(t)t^\kappa a_-(t), \quad (13)$$

where κ is an integer, and the factors a_\pm satisfy the following conditions:

- (1) $a_- \in L_p^-(\Gamma, \rho)$, $a_+ \in L_q^+(\Gamma, \rho^{1-q})$, $a_-^{-1} \in L_q^-(\Gamma, \rho^{1-q})$ and $a_+^{-1} \in L_p^+(\Gamma, \rho)$;
- (2) The operator $a_+^{-1} P a_-^{-1}$ is bounded in the space $L_p(\Gamma, \rho)$.

In the same works [5], [11], it is shown that the operator $aP + bQ$ is Noetherian in $L_p(\Gamma, \rho)$ if and only if $\text{ess inf } |a(t)| > 0$, $\text{ess inf } |b(t)| > 0$, and the function $c(t) = a(t)b^{-1}(t)$ admits a factorization $c(t) = c_+(t)t^\chi c_-(t)$. Under these conditions,

$$\text{Ind } A = -\chi.$$

The following theorem and the reasoning based on it will allow us to reduce the study of certain properties of singular operators in weighted spaces to the study of the corresponding properties of singular operators in unweighted spaces. Thus, in particular, we will conclude that it is sufficient to know Noetherian criteria for singular integral operators in unweighted L_p spaces.

Theorem 4.1. Let the function $h \in L_\infty(\Gamma)$ admit a factorization, $h = h_+ \cdot h_-$, in the space $L_p(\Gamma, \rho)$. We denote $L_p(\Gamma, \rho) = \mathcal{B}_1$ and $\mathcal{B}_2 = L_p(\Gamma, \rho \cdot |h_+|^{-p})$. There are two invertible operators $B \in L(\mathcal{B}_2, \mathcal{B}_1)$ and $F \in L(\mathcal{B}_1, \mathcal{B}_2)$ such that for any functions $a, b \in L_\infty(\Gamma)$ the equality holds

$$B(aP + bQ)F = haP + bQ. \quad (14)$$

Proof. We will define the operators F and B by the relations:

$$F = h_+ P + h_-^{-1} Q, \quad B = h_-. \quad (15)$$

We will prove that $B \in L(\mathcal{B}_2, \mathcal{B}_1)$ and $F \in L(\mathcal{B}_1, \mathcal{B}_2)$. Indeed, let $\varphi \in \mathcal{B}_2$, then

$$\begin{aligned} \|B\varphi\|_{\mathcal{B}_1}^p &= \int_\Gamma |h_-(t)|^p |\varphi(t)|^p \rho(t) |dt| = \int_\Gamma |\varphi(t)|^p |h(t)|^p |h_+(t)|^{-p} \rho(t) |dt| \\ &\leq \text{ess sup}_{t \in \Gamma} |h(t)|^p \|\varphi\|_{\mathcal{B}_2}^p. \end{aligned}$$

Therefore, $\|B\| \leq \text{ess sup}_{t \in \Gamma} |\omega(t)|$. We evaluate the norm of the operator F . Let $\psi \in \mathcal{B}_1$, then

$$\begin{aligned} \|F\psi\|_{\mathcal{B}_2}^p &= \int_\Gamma |\omega_+(t)(P\psi)(t) + \omega_-^{-1}(t)(Q\psi)(t)|^p |\omega_+(t)|^{-p} \rho(t) |dt| \\ &= \int_\Gamma |(P\psi)(t) + h(t)(Q\psi)(t)|^p \rho(t) |dt| = \|(P + hQ)\psi\|_{\mathcal{B}_1}^p. \end{aligned}$$

From here it follows that $\|F\| = \|P + hQ\|_{\mathcal{B}_1}$. The operators B and F are invertible, their inverses are the operators

$$B^{-1} = h_-^{-1}I, \quad F^{-1} = h_+^{-1}P + h_-Q.$$

The equality (14) itself is based on the properties of the projectors P and Q and is easily verified. Theorem 4.1 is proved. \square

From this theorem it follows

Corollary 4.1. *The operator $A = aP + bQ$ ($a, b \in L_\infty(\Gamma)$) is normally solvable in the space $L_p(\Gamma, \rho \cdot |h_+|^{-p})$ if and only if this property is also possessed in the space $L_p(\Gamma, \rho)$ by the operator*

$$A_h = haP + Q. \quad (16)$$

Corollary 4.2. *Under the conditions of Theorem 4.1, the following relations*

$$\dim \ker A|_{\mathcal{B}_2} = \dim \ker A_h|_{\mathcal{B}_1}, \quad \dim \ker A^*|_{\mathcal{B}_2^*} = \dim \ker A_h^*|_{\mathcal{B}_1^*}$$

hold.

Corollary 4.3. *The operator A admits a regularization (left, right or bilateral) in the space $L_p(\Gamma, \rho \cdot |h_+|^{-p})$ if and only if the operator A_h admits a similar regularization in the space $L_p(\Gamma, \rho)$. The regularizers M and M_h of the operators A and A_h are expressed by the relation*

$$M_h = FMB,$$

where B and F are the operators defined by equalities (15).

Corollary 4.4. *The operator A is invertible (left or right) in the space $L_p(\Gamma, \rho \cdot |h_+|^{-p})$ if and only if the operator A_h in the space $L_p(\Gamma, \rho)$ has this property. If A_h^{-1} is the inverse of the operator A_h , then the operator $FA_h^{-1}B$ is the inverse of the operator A .*

In what follows we will assume that the contour Γ consists of γ ($\gamma = 1, 2, \dots$) closed Liapunov lines on portions without common points (Figure 2).

Let

$$\rho(t) = \prod_{k=1}^n |t - t_k|^{\beta_k} \quad (-1 < \beta_k < p - 1).$$

We denote by $\Gamma_k(\subset \Gamma)$ the simple closed curve containing the point t_k and define the function

$$h_k(t) = \begin{cases} (t - z_k)^{-\varepsilon\beta_k/p}, & \text{for } t \in \Gamma_k, \\ 1, & \text{for } t \in \Gamma \setminus \Gamma_k, \end{cases} \quad (17)$$

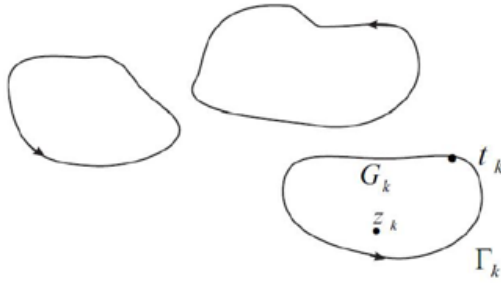


Figure 2. The composite contour Γ

where z_k is a point in the domain G_k , bounded by the curve Γ_k , $\varepsilon = 1$ ($\varepsilon = -1$) if the domain G_k , bounded by the oriented curve Γ_k , remains on the left (right), and $(t-z_k)^{-\varepsilon\beta_k/p}$ is the branch of the function $(z-z_k)^{-\varepsilon\beta_k}$ continuous at any point $t \in \Gamma \setminus \{t_k\}$ (Figure 2). The function $h_0(t) = h_1(t) \cdot h_2(t) \cdots h_\gamma(t)$ is continuous at any point $t \in \Gamma \setminus \{t_1, t_2, \dots, t_\gamma\}$ and

$$\frac{h_0(t_k - 0)}{h_0(t_k + 0)} = e^{-2\pi\beta_k/p}, \quad k = (1, 2, \dots, \gamma).$$

As

$$\frac{2\pi}{p} - 2\pi < -\frac{2\pi\beta_k}{p} < \frac{2\pi}{p},$$

from the results of [8], [11], [12], it follows that the operator $h_0P + Q$ is Noetherian in the space $L_p(\Gamma, \rho)$. Therefore, the function $h_0(t)$ admits a factorization in $L_p(\Gamma, \rho)$: $h_0 = h_+ \cdot h_-$. It is easy to show that

$$|h_+(t)|^{-p} = \rho(t) \cdot g(t),$$

where $(g(t))^{\pm 1} \in C(\Gamma)$. Therefore, $L_p(\Gamma, |h_+|^{-p}) = L_p(\Gamma, \rho)$. From these arguments and from Corollary 4.3 it follows

Theorem 4.2. *Let $a, b \in L_\infty(\Gamma)$ and let $h_0(t) = h_1(t) \cdot h_2(t) \cdots h_\gamma(t)$, where h_k are the functions defined by equality (17). The operator $A = aP + bQ$ is Noetherian in the space $L_p(\Gamma, \rho)$ if and only if the operator $A_h = haP + bQ$ is Noetherian in the space $L_p(\Gamma)$. Moreover, the following equalities hold:*

$$\dim \ker A|_{L_p(\Gamma, \rho)} = \dim \ker A_h|_{L_p(\Gamma)}, \quad \dim \ker A^*|_{L_q(\Gamma, \rho^{1-q})} = \dim \ker A_h^*|_{L_q(\Gamma)}.$$

Theorem 4.2 allows us to extend to the space L_p with weight ρ some results of I. Simonenko, A. Devinatz, H. Poisson, N. Rabindranathan, I. Daniliuc, N. Krupnik

which represent invertibility (or Noetherian) criteria of singular integral operators in the space L_p without weights.

We will focus on a result due to I. Simonenko. Let Γ consist of a finite number of Lyapunov-type curves without self-intersection points, and let $\rho(t) = \prod_{k=1}^n |t - t_k|^{\beta_k}$. From the results of I. Simonenko [5] with the help of Theorem 4.2, we obtain the following theorem.

Theorem 4.3. *Let the function $a \in L_\infty(\Gamma)$ satisfy the following three conditions:*

- (1) $\text{ess inf}_{t \in \Gamma} |a(t)| \neq 0$;
- (2) *for any point $t \in \Gamma \setminus \{t_1, \dots, t_n\}$ there is a neighborhood $u(\tau) \subset \Gamma$ and two functions g_τ^+, g_τ^- which belong to the sets $L_\infty^+(\Gamma)$, respectively $L_\infty^-(\Gamma)$ together with $(g_\tau^+)^{-1}$ and $(g_\tau^-)^{-1}$, such that, for $t \in u(\tau)$ the set of values of the function $g_\tau^+(t)a(t)g_\tau^-(t)$ is contained in the angle with the vertex at the point $z = 0$ and with the opening smaller than $\frac{2\pi}{\max(p,q)}$ ($p^{-1} + q^{-1} = 1$);*
- (3) *there is a neighborhood $u(t_k)$ of the point t_k ($k = 1, n$) and two functions g_k^+, g_k^- from the sets $L_\infty^+(\Gamma)$ and $L_\infty^-(\Gamma)$ together with $(g_k^+)^{-1}$ and $(g_k^-)^{-1}$, such that, for $t \in u_k(t_k)$ and $t > t_k$ ($t \in u(t_k)$, and $t < t_k$) the set of values of the function*

$$g_k^+(t)a(t)g_k^\tau(t) \quad (g_k^+(t)a(t)g_k^-(t) \cdot e^{-2\pi i\beta_k/p})$$

lies in the angle with the vertex at $z = 0$ and with the opening smaller than

$$\frac{2\pi}{\max(p, q)}.$$

Under these conditions the operator $aP + Q$ is Noetherian in the space $L_p(\Gamma, \rho)$.

We note that for $\rho(t) \equiv 1$ ($\beta_k = 0$, $k = 1, n$) this theorem is contained in the paper [5], and for $p = 2$ the conditions of Theorem 4.3 are necessary and sufficient in which the operator $aP + bQ$ is Noetherian. In addition, the results in the papers [5], [6] concerning the operators $aP + bQ$, where a and b are matrices of functions, can be extended using Theorem 4.3 to the space L_p^m with weights. Here we will not stop at these cases.

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