BOSE-EINSTEIM CONDENSATION OF TWO-DIMENSIONAL MAGNETOREXCITONS INTERACTING WITH PLASMONS UNDER THE INFLUENCE OF EXCITED LANDAU LEVELS: COLLECTIVE ELEMENTARY EXCITATIONS

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Abstract

The collective elementary excitations of two-dimensional magnetoexcitons in a Bose-Einstein condensate (BEC) with wave vector $\vec{k}=0$ were investigated in the framework of the Bogoliubov theory of quasiaverages. The Hamiltonian of the electrons and holes lying in the lowest Landau levels (LLLs) contains supplementary interactions due to virtual quantum transitions of the particles to the excited Landau levels (ELLs) and back. As a result, the interaction between the magnetoexcitons with $\vec{k}=0$ does not vanish and their BEC becomes stable. The equations of motion for the exciton operators $d(P)$ and $d^{\dagger}(P)$ are interconnected with equations of motion for the density operators $\rho(P)$ and $D(P)$. Instead of a set of two equations of motion, as in the case of usual Bose gas, corresponding to normal and abnormal Green's functions, we have a set of four equations of motion. Changing the center-of-mass wave vector of a magnetoexciton from 0 to $\vec{P}$, for example, implies changing its internal structure, because the internal distance between the Landau orbits of the quantized electron and hole becomes equal to $\left|\vec{P}\right|l^2$, where l is the magnetic length. The separated electrons and holes remaining in their Landau orbits can take part in the formation of magnetoexcitons as well as in the formation of collective plasma oscillations. These possibilities were not included in previous descriptions of the theory of structureless bosons or in the theory of Wannier-Mott excitons with a rigid relative electron-hole motion structure without the possibility of the intra-series excitations. The internal structure of magnetoexcitons is much less rigid than that of Wannier-Mott excitons, and the possibilities for electrons and holes to take part simultaneously in many processes are much more diverse. This means that we have to deal simultaneously with four branches of the energy spectrum, the two supplementary branches being the optical plasmon branch represented by the operator $\rho(P)$ and the acoustical plasmon branch represented by the operator $D(P)$. The energy spectrum of the collective elementary excitations consists of two exciton-type branches (energy and quasienergy branches), each of them with an energy gap and a

roton-type region, from the gapless optical plasmon branch and from the acoustical plasmon branch, which reveals an absolute instability in the range of small and intermediary wave vectors.

1. Introduction

A two-dimensional electron system in a strong perpendicular magnetic field reveals fascinating phenomena, such as the integer and fractional quantum Hall effects [1-5]. The discovery of the fractional quantum Hall effect (FQHE) fundamentally changed the established concepts about charged single-particle elementary excitations in solids [6-8]. Gauge transformations from the wave functions and creation operators of simple particles to other wave functions and creation operators describe composite particles (CPs) [9, 10] made from previous particles and quantum vortices created under the influence of the magnetic flux quanta [11-19]. Due to the contribution of many outstanding investigations [1-26] as well as many efforts to explain and represent the underlying processes in a clearer way [4, 27, 28], it is possible to make a short summary as follows. One can begin with the concept of composite particles proposed by Wilczek [9] in the form of particles with magnetic flux tubes attached. A posteriori, the flux tubes could be substituted by more concrete formations as point quantum vortices, as was argued by Read [11-14] in a series of papers. In many explanations proposed by Enger [4], it was underlined that in 3D space the particles may obey only Fermi and Bose statistics, whereas in 2D space fractional statistics are also possible, when under the interchanging of two particles, the wave function obtains the phase factor $e^{i\pi\alpha}$ with any fractional values of α. These particles were named “anyons” [9]. The gauge transformations [27] of the wave functions and of the creation and annihilation operators of the initial particles and the corresponding Hamiltonians was a powerful instrument revealing the fundamental physical processes hidden at the first sight in the quantum states of the system. For example, Halperin, Lee, and Read [15] investigated the FQHE for spinless electrons in a half-filled lowest Landau level (LLL) using a gauge transformation from the initial electron creation operator $\psi_e^\dagger(r)$ to another creation operator $\psi^\dagger(r)$ describing a more complicated object comprised of an electron and two vortices:

$$\psi^\dagger(r) = \psi_e^\dagger(r)\exp\left[-i\,\mathrm{m}\int d^2\vec{r}'\theta\left(\vec{r}-\vec{r}'\right)\hat{\rho}(\vec{r}')\right]; \qquad \theta\left(\vec{r}-\vec{r}'\right) = \arctan\frac{y-y'}{x-x'} \tag{1}$$

when the integer number *m* is even and equals 2. Here $\hat{\rho}(\vec{r}') = \psi_e^\dagger(r')\psi_e(r') = \psi^\dagger(r')\psi(r')$ and $\theta\left(\vec{r}-\vec{r}'\right)$ is the angle between the vector $\vec{r}-\vec{r}'$ and the axis x. It has a singular expression.

The transformation of type (1) must be accompanied by the transformation of the electromagnetic field $\vec{A}$, adding to it a gauge vector potential $\vec{a}$ with the components

$$a_i = -\frac{\hbar cm}{q}\partial_i \arctan\frac{y}{x} = \frac{\hbar cm}{q}\in^{ij}\partial_j \ln r$$

$$\vec{a} = \frac{\hbar cm}{q}\frac{\left(\vec{i}y-\vec{j}x\right)}{r^2} = \frac{\hbar cm}{qr}\vec{e}_\theta; \quad \vec{e}_\theta = \left(\vec{i}Sin\theta - \vec{j}Cos\theta\right) \tag{2}$$

Here q is the electric charge of the starting particles and $\in^{ij}$ is an antisymmetric tensor. Vector $\vec{a}$ coincides exactly with the velocity of a vortex with the strength κ

$$\vec{\mathrm{v}}(r) = \frac{\kappa\vec{e}_\theta}{2\pi r}. \tag{3}$$

This means that the gauge transformation revealed the existence of the vortices in the system created by the magnetic flux quanta. Vortex (2) has a potential flow almost everywhere excluding the point $\vec{r}=0$. The "magnetic" field strength $b(\vec{r})$ created by the gauge Chern-Simons potential $a(\vec{r})$ equals [4, 24, 27]

$$b(r)=Curl\vec{a}(r)=\in^{ij}\partial_i a_j=-\frac{\hbar cm}{q}\Delta\ln r=-\phi_0 m\delta^{(2)}(\vec{r})Sgn(q)$$
$$\phi_0=\frac{2\pi\hbar c}{|q|};\qquad \frac{\Delta\ln r}{2\pi}=\delta^{(2)}(\vec{r});\qquad q=|q|Sgn(q);\ \int b(r)d^2\vec{r}=\oint\vec{a}d\vec{l}=-m\phi_0 Sgn(q) \tag{4}$$

The "magnetic" field strength has also a singularity, being different from zero only at the point $\vec{r}=0$. But the "magnetic" field flux is different from zero despite the fact that the vortex area is zero. These singular vortices are called point vortices.

Girvin, MacDonald and Platzman [19] elaborated the theory of the collective elementary excitation spectrum in the case of the FQHE, closely analogous to Feynman's theory of superfluid helium. The predicted spectrum has a gap at $\vec{k}=0$ and a deep magneto-roton minimum at finite wavevector, which is a precursor to the gap collapse associated with Wigner crystal instability.

In this paper, we study a coplanar electron-hole (e-h) system with electrons in a conduction band and holes in a valence band, both of which have Landau levels in a strong perpendicular magnetic field. Earlier, this system has been studied in a series of papers [29-37] mostly dedicated to the theory of 2D magnetoexcitons. This system bears some resemblance to the case of a bilayer electron system with only conduction band occupied, in which the Landau level in each of two adjacent 2D layers is half occupied, adding free charge in a thin metallic layer grown nearly but separated from the bilayer. The system may be described as a having Landau levels half occupied by holes in one layer and half occupied by electrons in another layer. The ground state of this system may be viewed as a Bose-Einstein condensation, or more properly a BCS state, of electron-hole pairs, i.e., excitons at high density [38-43]. The system we are interested in has only one layer, with electrons in conduction band and holes in the valence band of the same layer created by optical excitation or by p-n doping injection (both of these methods can be called "pumping"). In this case, there is an intrinsic metastability, since electrons in the conduction band can drop down into the valence band and recombine with holes there. However, we assume that the recombination rate of the electrons with holes has such a slow rate that the number of electrons and holes is nearly conserved. Unlike the case of the bilayer electron system with a half-filled lowest Landau level, in the case of a single excited layer which we consider, the density of excitons can be quite low, so that the electron Landau level and the separate hole Landau level are each only slightly occupied, and Pauli exclusion and phase space filling do not come in to play.

We are interested in the distribution of the flux quanta in the case of an electron-hole system with equal average numbers of electrons and holes $\bar{N}_e=\bar{N}_h$ with filling factor $\nu=\bar{N}_e/N$, where N is the total number of flux quanta. In the case of fractional integer filling factor, there are an integer number of flux quanta per each e-h pair. The creation of the vortices in this case is not studied at the present time.

Another related system is the case of a pumped bilayer system [44-54], such as coupled semiconductor quantum wells (CQWs). While the geometry of this system with two adjacent layers is similar to the bilayer electron system discussed above, it is actually more similar in its main properties to the pumped single-layer electron-hole system which is the topic of this paper. In this system, the main effect of the bilayer structure is to put a tunneling barrier between the electrons which are in one layer and the holes which are in the other layer. This makes it possible to experimentally realize the condition given above of negligible electron-hole recombination; experimentally, exciton lifetimes of up to 40 ms have been observed [52, 54], and the long lifetimes allow diffusion of the excitons over macroscopic distances [44] and equilibration to the lattice temperature in a harmonic potential [45]. Like the pumped single-layer system discussed here, the electrons in the conduction band and the holes in the valence band can each be at very low density. One drawback of this system is that it causes the interactions of the excitons to be enhanced, due to their alignment in the direction perpendicular to the layers, giving a strong dipole-dipole interaction [46]. For a comparison of the half-filled bilayer system and the pumped bilayer system, see [54].

In this bilayer system with nearly negligible recombination, a magnetic field can be applied in the direction perpendicular to the layers to create magnetoexcitons of the type considered here. So far, there has been no evidence of Bose-Einstein condensation in this system. The magnetoexciton regime can be reached, however, as shown in Fig. 1. At high magnetic field, the exciton energy shifts up to higher energy following the Landau level energy, as expected. The fast shift to lower energy seen at low magnetic field has been explained as due to the effect of the disorder on the tunneling current in the samples [47].

Recently, another group has used magnetic field to study the Mott transition from insulating exciton gas to conducting plasma [48]. They were able to show that the magnetic response of the system changed sharply when the system had undergone a Mott transition. The Mott transition of exciton gas to plasma is in general still a quite difficult problem receiving much study [49, 50].

One reason why BEC has not been observed clearly in either the simple pumped bilayer system or the pumped magnetoexciton bilayer system may be that the condensate occurs in a "dark" state which does not emit light. It has been proposed [55-57] that condensation of excitons will always occur in a dark state if one exists; some evidence of BEC of dark excitons has been reported in [52]. The existence of two types of excitons—dark and bright excitons—is related to their spin structure. In GaAs, the lowest exciton state splits into a J=2 doublet which does not emit light, by symmetry, and a J=1 doublet which emits light. In the present paper, we do not take into account the spin structure of the excitons.

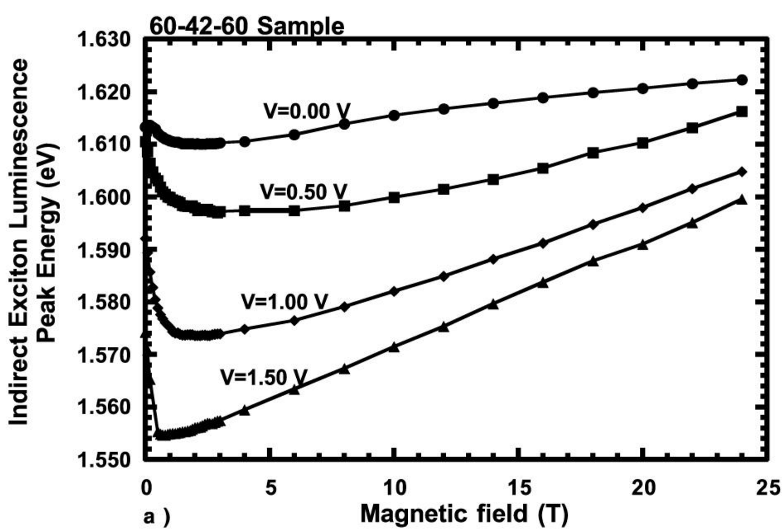

Fig. 1. Energy of the indirect exciton photon emission as a function of magnetic field, for several values of the electric field (from [45]).

Our paper is organized as follows. In Section 2, we deduce the Hamiltonian of the supplementary interactions between electrons and holes lying in the lowest Landau levels (LLLs) due to their virtual quantum transition from the LLLs to excited Landau levels (ELLs) in the processes of Coulomb scattering. It is interesting that this interaction due to the influence of the ELLs is possible to express in terms of two-particle operators in the form of density fluctuation operators. In Section 3, the full Hamiltonian containing the basic Coulomb interaction as well as the supplementary interaction is taken through the operation of gauge symmetry breaking so as to study the Bose-Einstein condensation (BEC) of magnetoexcitons. The traditional way is to follow the Keldysh–Kozlov–Kopaev [58] method describing the BEC of excitons in the electron-hole representation, which leads to the subsequent Bogoliubov u-v transformation of the creation and annihilation operators of the initial and final Fermi-type quasiparticles. In the present paper, we have chosen another variant proposed by Bogoliubov in his theory of quasiaverages [59]. Both methods are equivalent, but the second variant is preferable for our purposes, because we will operate with the integral two-particle operators rather than with single-particle Fermi-type operators. In Section 4, we deduce the equations of motion for the integral two-particle operators and for the corresponding Green's functions. The truncation of the equations of motion permit us to obtain a Dyson equation in a 4×4 matrix form and to determine the self-energy parts $\Sigma_{i,j}$ with $i,j=1,2,3,4$. Section 5 is devoted to the analytical and numerical calculations of the energy spectrum in different approximations. The conclusions are presented in Section 6.

2. Hamiltonian of the supplementary interaction

The Hamiltonian of the Coulomb interaction of the electrons and holes lying on their LLLs has the form

$$H_c=\frac{1}{2}\sum_{\vec{Q}} W_{\vec{Q}}\left[\rho(\vec{Q})\rho(-\vec{Q})-N_e-N_h\right]-\mu_e N_e-\mu_h N_h \tag{5}$$

Here $\rho(\vec{Q})$ are the density fluctuation operators expressed through the electron $\rho_e(\vec{Q})$ and hole $\rho_h(\vec{Q})$ density operators as follows

$$\begin{gathered}\rho_e(\vec{Q})=\sum_t e^{iQ_y t l^2} a^{\dagger}_{t-\frac{Q_x}{2}} a_{t+\frac{Q_x}{2}};\;\; \rho_h(\vec{Q})=\sum_t e^{iQ_y t l^2} b^{\dagger}_{t+\frac{Q_x}{2}} b_{t-\frac{Q_x}{2}}\\ \rho(\vec{Q})=\rho_e(\vec{Q})-\rho_h(-\vec{Q});\;\; D(\vec{Q})=\rho_e(\vec{Q})+\rho_h(-\vec{Q});\\ N_e=\rho_e(0);\;\; N_h=\rho_h(0);\;\; N=N_e+N_h;\;\; W_{\vec{Q}}=\frac{2\pi e^2}{\varepsilon_0 S|\vec{Q}|}e^{-Q^2l^2/2}\end{gathered} \tag{6}$$

The density operators are integral two-particle operators. They are expressed through the single-particle creation and annihilation operators $a_p^{\dagger}, a_p$ for electrons and $b_p^{\dagger}, b_p$ for holes. ε_0 is the dielectric constant of the background; μ_e and μ_h are chemical potentials for electrons and holes.

The supplementary indirect interactions between electrons and holes appear due to the simultaneous virtual quantum transitions of two particles from the LLLs to excited Landau levels (ELLs) and their return back during the Coulomb scattering processes. This interaction was deduced in [37]. It has a general attractive character and has the form

$$
\begin{aligned}
H_{\mathrm{suppl}} = & -\frac{1}{2}\sum_{p,q,s} \phi_{e-e}(p,q;s) a_p^\dagger a_q^\dagger a_{q+s} a_{p-s} - \\
& -\frac{1}{2}\sum_{p,q,s} \phi_{h-h}(p,q;s) b_p^\dagger b_q^\dagger b_{q+s} b_{p-s} - \sum_{p,q,s} \phi_{e-h}(p,q;s) a_p^\dagger b_q^\dagger b_{q+s} a_{p-s}
\end{aligned}
\tag{7}
$$

An important property of this quartic form constructed from single-particle operators is the possibility to transcribe it through the integral two-particle operators $\hat{\rho}(\vec{Q})$ and $\hat{D}(\vec{Q})$ as follows [60]:

$$
H_{\mathrm{suppl}} = \frac{1}{2} B_{i-i} N - \frac{1}{4N}\sum_{Q} V(Q)\rho(\vec{Q})\rho(-\vec{Q}) - \frac{1}{4N}\sum_{Q} U(Q) D(\vec{Q}) D(-\vec{Q}) \tag{8}
$$

Paper [60] contains a detailed derivation of the Dyson equation, but the approach of solution is completely different.
The estimates give the values [37, 60]

$$
\begin{gathered}
U(\vec{Q}) \cong U(0) e^{-Q^2 l^2/2}; \;\; U(0) = 2A_{i-i}; \;\; \frac{1}{N}\sum_{\vec{Q}} U(\vec{Q}) = B_{i-i} + \Delta(0); \\
V(\vec{Q}) \approx V(0) = 0; \;\; A_{i-i} = 0.481\frac{I_l^2}{\pi\hbar\omega_c}; \;\; B_{i-i} = 0.432\frac{I_l^2}{\pi\hbar\omega_c}; \;\; \Delta(0) = 0.688\frac{I_l^2}{\pi\hbar\omega_c}
\end{gathered}
\tag{9}
$$

Here I_l is the ionization potential of magnetoexcitons and $\hbar\omega_c$ is the cyclotron frequency at $m_e = m_h$. The full Hamiltonian describing the interaction of electrons and holes lying on the LLLs is

$$
H = H_{Coul} + H_{\mathrm{suppl}} \tag{10}
$$

3. Breaking of the gauge symmetry. Bose-Einstein condensation of magnetoexcitons with k=0

In [30-36], Bose-Einstein condensation (BEC) of magnetoexcitons with wave vector $\vec{k}$ different from zero was considered without taking into account the influence of the excited Landau levels (ELLs). However, the case of BEC with $\vec{k} = 0$ was impossible to incorporate in the previous description because these magnetoexcitons form an ideal Bose gas: in that model, the interaction between two magnetoexcitons with electrons and holes lying on the LLLs and with $\vec{k} = 0$ equals exactly zero. This prediction of ideal behavior is an artifact of the assumptions of the theory and made approximations, however. To consider BEC of magnetoexcitons with $\vec{k} = 0$, it is necessary to take into account the influence of the ELLs, represented by Hamiltonian (8).

As discussed in previous papers [31-37, 61], the breaking of the gauge symmetry of Hamiltonian (10) can be achieved using the Keldysh–Kozlov–Kopaev [58] method using the unitary transformation

$$
D(\sqrt{N_{ex}}) = \exp[\sqrt{N_{ex}}(d^\dagger(\vec{k}) - d(\vec{k}))] \tag{11}
$$

where $d^\dagger(k)$ and $d(k)$ are the creation and annihilation operators of the magnetoexcitons. In the electron-hole representation they are [31-37]:

$$d^{\dagger}(\vec{P}) = \frac{1}{\sqrt{N}} \sum_{t} e^{-iP_y t l^2} a^{\dagger}_{t+\frac{P_x}{2}} b^{\dagger}_{-t+\frac{P_x}{2}};$$
$$d(\vec{P}) = \frac{1}{\sqrt{N}} \sum_{t} e^{iP_y t l^2} b_{-t+\frac{P_x}{2}} a_{t+\frac{P_x}{2}}; \tag{12}$$

BEC of magnetoexcitons leads to the formation of a coherent macroscopic state as a ground state of the system with wave function

$$\left|\psi_g(\vec{k})\right\rangle = \hat{D}(\sqrt{N_{ex}})\left|0\right\rangle; \quad a_p\left|0\right\rangle = b_p\left|0\right\rangle = 0 \tag{13}$$

Here $\left|0\right\rangle$ is the vacuum state for electrons and holes. In spite of the fact that we have kept an arbitrary value of $\vec{k}$, nevertheless our main goal is the BEC with $\vec{k}=0$. The transformed Hamiltonian (10) looks like

$$\mathcal{H} = D\left(\sqrt{N_{ex}}\right) H D^{\dagger}\left(\sqrt{N_{ex}}\right) \tag{14}$$

and is succeeded, as usual, by the Bogoliubov *u*-*v* transformations of the single-particle Fermi operators

$$\alpha_p = D\left(\sqrt{N_{ex}}\right) a_p D^{\dagger}\left(\sqrt{N_{ex}}\right) = u a_p - \mathrm{v}(p - \frac{k_x}{2}) b^{\dagger}_{k_x - p}; \quad \alpha_p\left|\psi_g(\vec{k})\right\rangle = 0$$
$$\beta_p = D\left(\sqrt{N_{ex}}\right) b_p D^{\dagger}\left(\sqrt{N_{ex}}\right) = u b_p + \mathrm{v}(\frac{k_x}{2} - p) a^{\dagger}_{k_x - p}; \quad \beta_p\left|\psi_g(\vec{k})\right\rangle = 0 \tag{15}$$

Instead of this traditional way of transforming the expressions of the starting Hamiltonian (10) and of the integral two-particle operators (6) and (12), we will use the method proposed by Bogoliubov in his theory of quasiaverages [59], remaining in the framework of the original operators. The new variant is completely equivalent to the previous one, and both of them can be used in different stages of the calculations. For example, the average values can be calculated using wave function (13) and *u*-*v* transformations (15), whereas the equations of motion for the integral two-particle operators can be simply written in the starting representation.

Hamiltonian (10) with the broken gauge symmetry in the lowest approximation has the form

$$\mathcal{H} = \frac{1}{2}\sum_{\vec{Q}} W_{\vec{Q}}\left[\rho(\vec{Q})\rho(-\vec{Q}) - N_e - N_h\right] - \mu_e N_e - \mu_h N_h +$$
$$+\frac{1}{2}B_{i-i}N - \frac{1}{4N}\sum_{Q} V(Q)\rho(\vec{Q})\rho(-\vec{Q}) - \frac{1}{4N}\sum_{Q} U(Q)D(\vec{Q})D(-\vec{Q}) - \tag{16}$$
$$-\tilde{\eta}\sqrt{N}\left(d^{\dagger}(k) + d(k)\right)$$

Another smaller term, proportional to $\tilde{\eta}$, has been dropped for simplicity. Here the parameter $\tilde{\eta}$, which determines the breaking of the gauge symmetry, depends, as in the case of weakly non-ideal Bose-gas considered by Bogoliubov [59], on the chemical potential μ and on the square root of the density. In our case, the density is proportional to the filling factor $\nu = \mathrm{v}^2$. We have

$$\mu = \mu_e + \mu_h; \quad \bar{\mu} = \mu + I_l; \quad N_{ex} = \mathrm{v}^2 N; \quad \tilde{E}_{ex}(k) = -I_l - \Delta(k) + E(k);$$
$$\tilde{\eta} = (\tilde{E}_{ex}(k) - \mu)\mathrm{v} = (E(k) - \Delta(k) - \bar{\mu})\mathrm{v}; \qquad E(k) = 2\sum_{Q} W_Q Sin^2\left(\frac{[K\times Q]_z l^2}{2}\right); \tag{17}$$

In the special case of $\vec{k}=0$ we obtain

$$\tilde{\eta}=-(\bar{\mu}+\Delta(0))\mathrm{v} \tag{18}$$

4. Equations of motion for the integral two-particle operators. Green's functions, Dyson equation and self-energy parts

The equations of motion for the integral two-particle operators with wave vectors $\vec{P}\neq 0$ in the special case of BEC of magnetoexcitons with $\vec{k}=0$ are

$$\begin{aligned}
& i\hbar\frac{d}{dt}d(\vec{P})=[d(\vec{P}),\hat{\mathcal{H}}]=(-\bar{\mu}+E(\vec{P})-\Delta(\vec{P}))d(\vec{P})-2i\sum_{\vec{Q}}\tilde{W}(\vec{Q})Sin\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)\rho(\vec{Q})d(\vec{P}-\vec{Q})- \\
& -\frac{1}{N}\sum_{\vec{Q}}U(\vec{Q})Cos\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)D(\vec{Q})d(\vec{P}-\vec{Q})+\tilde{\eta}\frac{D(\vec{P})}{\sqrt{N}}; \\
& i\hbar\frac{d}{dt}d^\dagger(-\vec{P})=[d^\dagger(-\vec{P}),\hat{\mathcal{H}}]=(\bar{\mu}-E(-\vec{P})+\Delta(-\vec{P}))d^\dagger(-\vec{P})+ \\
& +2i\sum_{\vec{Q}}\tilde{W}(\vec{Q})Sin\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)d^\dagger(-\vec{P}-\vec{Q})\hat{\rho}(-\vec{Q})+ \\
& +\frac{1}{N}\sum_{\vec{Q}}U(\vec{Q})Cos\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)d^\dagger(-\vec{P}-\vec{Q})D(-\vec{Q})-\tilde{\eta}\frac{D(\vec{P})}{\sqrt{N}}; \\
& \quad i\hbar\frac{d}{dt}\rho(\vec{P})=[\rho(\vec{P}),\hat{\mathcal{H}}]= \\
& \quad =-i\sum_{\vec{Q}}\tilde{W}(\vec{Q})Sin\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)[\rho(\vec{P}-\vec{Q})\rho(\vec{Q})+\rho(\vec{Q})\rho(\vec{P}-\vec{Q})]+ \\
& \quad +\frac{i}{2N}\sum_{\vec{Q}}U(\vec{Q})Sin\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)\left[D(\vec{P}-\vec{Q})D(\vec{Q})+D(\vec{Q})D(\vec{P}-\vec{Q})\right]; \\
& \quad i\hbar\frac{d}{dt}\hat{D}(\vec{P})=[\hat{D}(\vec{P}),\hat{\mathcal{H}}]= \\
& \quad -i\sum_{\vec{Q}}\tilde{W}(\vec{Q})Sin\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)[\rho(\vec{Q})D(\vec{P}-\vec{Q})+D(\vec{P}-\vec{Q})\rho(\vec{Q})]+ \\
& \quad +\frac{i}{2N}\sum_{\vec{Q}}U(\vec{Q})Sin\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)[D(\vec{Q})\rho(\vec{P}-\vec{Q})+\rho(\vec{P}-\vec{Q})D(\vec{Q})]+2\tilde{\eta}\sqrt{N}\left[d(\vec{P})-d^\dagger(-\vec{P})\right];
\end{aligned} \tag{19}$$

Following equations of motion (19), we introduce four interconnected retarded Green's functions at $T=0$ [62, 63]:

$$\begin{aligned}
& G_{11}(\vec{P},t)=\left\langle\left\langle d(\vec{P},t);X^\dagger(\vec{P},0)\right\rangle\right\rangle; \quad G_{12}(\vec{P},t)=\left\langle\left\langle d^\dagger(-\vec{P},t);X^\dagger(\vec{P},0)\right\rangle\right\rangle; \\
& G_{13}(\vec{P},t)=\left\langle\left\langle\frac{\rho(\vec{P},t)}{\sqrt{N}};X^\dagger(\vec{P},0)\right\rangle\right\rangle; \; G_{14}(\vec{P},t)=\left\langle\left\langle\frac{D(\vec{P},t)}{\sqrt{N}};X^\dagger(\vec{P},0)\right\rangle\right\rangle;
\end{aligned} \tag{20}$$

as well as their Fourier transforms $G_{ij}(\vec{P},\omega)$, for which the equations of motion of the same type as the equations of motion (19) were obtained. These Green's functions can be called one-operator Green's functions, because they contain only one two-particle operator of the type $d^\dagger$, d, ρ, D. However, on the right side of the corresponding equations of motion, there is a second generation of two-operator Green's functions containing the different products of the two-particle operators mentioned above. For them, the second generation of the equations of motion was deduced, containing in their right sides the Green's function of the third generation. They are three-operator Green's functions for which it is necessary to deduce the third generation of equations of motion. But we have stopped here the evolution of the infinite chains of equations of motion for multi-operators Green's function following the procedure proposed by Zubarev [63]. The truncation of the chains of the equations of motion and the decoupling of the one-operator Green's functions from the multi-operator Green's functions was achieved by substituting the three-operator Green's functions by one-operator Green's functions multiplied by the average value of the remaining two operators. The average values were calculated using the ground state wave function (13) and the *u*-*v* transformations (15). The Zubarev procedure is equivalent to a perturbation theory with a small parameter of the type $\mathrm{v}^2(1-\mathrm{v}^2)$, which represents the product of a filling factor $\nu = \mathrm{v}^2$ and the phase-space filling factor $(1-\mathrm{v}^2)$ reflecting the Pauli exclusion principle.

The closed system of Dyson equations has the form

$$\sum_{j=1}^{4} G_{1j}(\vec{P},\omega)\Sigma_{jk}(\vec{P},\omega) = C_{1k}; \qquad k = 1,2,3,4 \tag{21}$$

There are 16 different components of the self-energy parts $\Sigma_{jk}(\vec{P},\omega)$ forming a 4×4 matrix. They are

$$\Sigma_{11}(\vec{P},\omega) = \hbar\omega + i\delta + \bar{\mu} - E(\vec{P}) + \Delta(\vec{P}) - \frac{\langle D(A)D(-A)\rangle}{N^2}\sum_{\vec{Q}\neq\vec{P}} \frac{U^2(\vec{Q})Cos^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)}{\hbar\omega + i\delta + \bar{\mu} - E(\vec{P}-\vec{Q}) + \Delta(\vec{P}-\vec{Q})};$$

$$\Sigma_{21}(\vec{P},\omega) = 0;$$

$$\Sigma_{31}(\vec{P},\omega) = i\frac{\langle D(A)d(-A)\sqrt{N}\rangle}{N^2}\sum_{\vec{Q}\neq\vec{P}} \frac{U(\vec{Q})U(\vec{Q}-\vec{P})Cos\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)Sin\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)}{\hbar\omega + i\delta + \bar{\mu} - E(\vec{P}-\vec{Q}) + \Delta(\vec{P}-\vec{Q})};$$

$$\Sigma_{41}(\vec{P},\omega) = -\tilde{\eta} + U(\vec{P})\frac{\langle d(0)\rangle}{\sqrt{N}} + 2\frac{\langle D(A)d(-A)\sqrt{N}\rangle}{N}\sum_{\vec{Q}\neq\vec{P}} \frac{\tilde{W}(\vec{Q})(U(\vec{P}) - U(\vec{Q}-\vec{P}))Sin^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)}{\hbar\omega + i\delta + \bar{\mu} - E(\vec{P}-\vec{Q}) + \Delta(\vec{P}-\vec{Q})} -$$
$$- \frac{\langle D(A)d(-A)\sqrt{N}\rangle}{N^2}\sum_{\vec{Q}\neq\vec{P}} \frac{U(\vec{Q})U(\vec{P})Cos^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)}{\hbar\omega + i\delta + \bar{\mu} - E(\vec{P}-\vec{Q}) + \Delta(\vec{P}-\vec{Q})};$$

$$\sum\nolimits_{12}(\vec{P},\omega)=0;$$

$$\sum\nolimits_{22}(\vec{P},\omega)=\hbar\omega+i\delta-\bar{\mu}+E(\vec{P})-\Delta(-\vec{P})-\frac{\langle D(A)D(-A)\rangle}{N^2}\sum_{\vec{Q}\neq-\vec{P}}\frac{U(\vec{Q})U(-\vec{Q})Cos^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)}{\hbar\omega+i\delta-\bar{\mu}+E(-\vec{P}-\vec{Q})-\Delta(-\vec{P}-\vec{Q})};$$

$$\sum\nolimits_{32}(\vec{P},\omega)=i\frac{\langle d^\dagger(A)D(-A)\sqrt{N}\rangle}{N^2}\sum_{\vec{Q}\neq-\vec{P}}\frac{U(\vec{Q})U(-\vec{Q}-\vec{P})Cos\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)Sin\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)}{\hbar\omega+i\delta-\bar{\mu}+E(-\vec{P}-\vec{Q})-\Delta(-\vec{P}-\vec{Q})};$$

$$\sum\nolimits_{42}(\vec{P},\omega)=\tilde{\eta}-U(-\vec{P})\frac{\langle d^\dagger(0)\rangle}{\sqrt{N}}-2\frac{\langle d^\dagger(A)D(-A)\sqrt{N}\rangle}{N}\sum_{\vec{Q}\neq-\vec{P}}\frac{\tilde{W}(\vec{Q})(U(-\vec{Q}-\vec{P})-U(\vec{P}))Sin^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)}{\hbar\omega+i\delta-\bar{\mu}+E(-\vec{P}-\vec{Q})-\Delta(-\vec{P}-\vec{Q})}-$$
$$-\frac{\langle d^\dagger(A)D(-A)\sqrt{N}\rangle}{N^2}\sum_{\vec{Q}\neq-\vec{P}}\frac{U(\vec{Q})U(-\vec{P})Cos^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)}{\hbar\omega+i\delta-\bar{\mu}+E(-\vec{P}-\vec{Q})-\Delta(-\vec{P}-\vec{Q})}; \quad (22)$$

$$\sum\nolimits_{13}(\vec{P},\omega)=0;$$
$$\sum\nolimits_{23}(\vec{P},\omega)=0;$$
$$\sum\nolimits_{33}(\vec{P},\omega)=\hbar\omega+i\delta-\frac{\langle D(A)D(-A)\rangle}{N^2(\hbar\omega+i\delta)}\sum_{\vec{Q}\neq\vec{P}}U(\vec{Q})(U(-\vec{Q})-U(\vec{Q}-\vec{P}))Sin^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right);$$
$$\sum\nolimits_{43}(\vec{P},\omega)=0;$$
$$\sum\nolimits_{14}(\vec{P},\omega)=-2\tilde{\eta};$$
$$\sum\nolimits_{24}(\vec{P},\omega)=2\tilde{\eta};$$
$$\sum\nolimits_{34}(\vec{P},\omega)=0;$$
$$\sum\nolimits_{44}(\vec{P},\omega)=\hbar\omega+i\delta-\frac{2\langle D(A)D(-A)\rangle}{N(\hbar\omega+i\delta)}\sum_{\vec{Q}\neq-\vec{P}}\tilde{W}(\vec{Q})(U(\vec{Q}-\vec{P})-U(\vec{P}))Sin^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right)+$$
$$+\frac{\langle D(A)D(-A)\rangle}{N^2(\hbar\omega+i\delta)}\sum_{\vec{Q}\neq\vec{P}}U(\vec{Q})(U(\vec{P})-U(-\vec{Q}))Sin^2\left(\frac{[\vec{P}\times\vec{Q}]l^2}{2}\right);$$

5. Energy spectrum of the collective elementary excitations

The self-energy parts (22) contain the average values of two-operator products, which appeared after the decoupling of the three-operator Green's functions by expressing them through one-operator Green's functions. The average values were calculated using the ground state wave function (13) and the coefficients of the u-v transformation (15). For operators $\hat{D}(\vec{Q})$, $\hat{\rho}(\vec{Q})$, $d(\vec{Q})$ and $d^\dagger(\vec{Q})$ with $\vec{Q}\neq 0$ we have

$$\begin{aligned}
&\left\langle D(\vec{Q})D(-\vec{Q})\right\rangle = 4u^2\mathrm{v}^2 N;\\
&\left\langle \rho(\vec{Q})\rho(-\vec{Q})\right\rangle = 0;\\
&\left\langle D(\vec{Q})d(-\vec{Q})\right\rangle = \left\langle d^{\dagger}(\vec{Q})D(-\vec{Q})\right\rangle = -2u\mathrm{v}\sqrt{N};\\
&\left\langle \rho(\vec{Q})d(-\vec{Q})\right\rangle = \left\langle d^{\dagger}(\vec{Q})\rho(-\vec{Q})\right\rangle = 0\\
&\left\langle d(0)\right\rangle = \left\langle d^{\dagger}(0)\right\rangle = u\mathrm{v}\sqrt{N};\\
&u^2+\mathrm{v}^2=1; \quad N=\frac{S}{2\pi l^2}
\end{aligned} \tag{23}$$

Averages (23) have extensive values depending on N. For the condensate with $\vec{k}=0$, they do not depend on the wave vector $\vec{Q}$. At the point $\vec{Q}=0$ their values do not coincide with (15), changing by a jump. Expressions (23) have a small parameter of the perturbation theory in the forms $u^2\mathrm{v}^2=\mathrm{v}^2(1-\mathrm{v}^2)$, $u\mathrm{v}^3$ and $u\mathrm{v}$. The chemical potential $\bar{\mu}$ and the parameter $\tilde{\eta}$ of the quasiaverage theory are

$$\begin{aligned}
&\bar{\mu}+\Delta(0)=2\mathrm{v}^2(B_{i-i}-2A_{i-i}+\Delta(0))=0.1rI_l\mathrm{v}^2\\
&\tilde{\eta}=-0.1rI_l\mathrm{v}^3;\\
&0<r=\frac{I_l}{\hbar\omega_c}\le 1; \quad I_l=\frac{e^2}{\varepsilon_0 l}\sqrt{\frac{\pi}{2}}
\end{aligned} \tag{24}$$

Here l is the magnetic length, ε_0 is the dielectric constant of the background and I_l is the ionization potential of the magnetoexciton. The chemical potential $\bar{\mu}$ has an increasing dependence on the concentration of electrons n_e or of magnetoexcitons n_{ex}, which are proportional to the filling factor $\nu=\mathrm{v}^2$ as follows

$$n_e=n_{ex}=\frac{\mathrm{v}^2}{2\pi l^2} \tag{25}$$

This means that the system of Bose-Einstein condensed magnetoexcitons with wave vector $\vec{k}=0$ is stable against collapse. Its stability is completely due to the influence of the ELLs. In spite of the fact that the supplementary interaction has an overall attractive character and its average values in the Hartree approximation remain attractive; nevertheless, the exchange-type Fock terms as well as other terms arising due to the Bogoliubov *u-v* transformation give rise to a repulsive interaction in the system, which is necessary to stabilize the BEC of magnetoexcitons with $\vec{k}=0$. The role of the exchange Fock terms supplying a repulsion in conditions of overall attraction is similar to the case of the electron plasma, when the exchange Fock terms supply an effective attraction and lowering of the energy per particle in the condition of an overall Coulomb repulsion between the electrons [64]. The influence of the exchange Fock terms is the same in the case of electron-hole liquid (EHL) [65]. Another system with mixed interactions, the Wannier-Mott excitons with different spin projections, was investigated in [66]. It was shown that the coherent pairing of excitons leads to Bose-Einstein condensation of biexcitons. There the excitons with opposite spin projections and attractive interaction formed the biexcitons, whereas other excitons with parallel spin projections and repulsive interactions stabilized the Bose-Einstein condensate of biexcitons. The situation with magnetoexcitons also does not coincide exactly with the Bogoliubov model of a weakly non-ideal Bose gas with pure repulsive interaction, because the presence of the attractive Hartree terms mentioned above.

This difference will be reflected in the structure of the energy spectrum of the collective elementary excitations. The collective elementary excitation spectrum obtained by Bogoliubov [59] is gapless. It has a linear dependence on the wave vector in the range of long wavelengths and a quadratic dependence in another range of spectrum. Similar results were obtained by Beliaev [67] as well as by Keldysh and Kozlov [58], who first proposed the theory of the BEC of Wannier-Mott excitons in the electrons-hole description.

The influence of the attractive Hartree terms will generate the gap in the energy spectrum of the exciton branches of the collective elementary excitations, which will be studied below. The influence of the ELLs and their stabilizing role in the theory of the BEC of magnetoexcitons with nonzero wave vector $\vec{k}$ quickly decreases with increasing kl. In the range of $kl \geq 0.5$, the Bose–Einstein condensate becomes unstable in the Hartree–Fock–Bogoliubov approximation [37]. Only in the range of $kl \sim 3-4$ does the ability to stabilize the condensate appear, taking into account the Anderson-type coherent excited states and the correlation energy calculated on this base. Under these conditions, a metastable dielectric liquid phase (MDLP) formed by Bose-Einstein condensed magnetoexcitons with kl ≈ 3-4 was found [35]. The collective elementary excitations under these conditions were investigated in [61].

In the case of $k=0$, the self-energy parts (22) contain only the coefficients linear in $U(P)$, or quadratic dependences of the types $U(Q)U(-Q)$ and $\tilde{W}(Q)U(-Q-P)$ which reflect the influence of the ELLs. The equations of motion for the exciton operators $d(P)$ and $d^{\dagger}(-P)$ are interconnected with equations of motion for the density operators $\rho(P)$ and $D(P)$. Instead of a set of two equations of motion, as in the case of the usual Bose gas corresponding to normal and abnormal Green's functions, we have a set of four equations of motion. Changing the center-of-mass wave vector of the magnetoexciton, for example, from 0 to $\vec{P}$, means changing its internal structure, because the internal distance between the Landau orbits of the quantized electron and hole becomes equal to $\left|\vec{P}\right| l^2$. The separated electrons and holes remaining in their Landau orbits can take part in the formation of magnetoexcitons as well as in collective plasma oscillations. These possibilities were not considered in the theory of structureless bosons or in the case of Wannier-Mott excitons with a rigid relative electron-hole motion structure without the possibility of the intra-series excitations. In the case of magnetoxcitons, their internal structure is much less rigid than in the case of Wannier-Mott excitons and the possibilities for electrons and holes to take part simultaneously in many processes are much more diverse. Instead of the branches of the energy spectrum corresponding to normal and abnormal Green's functions we have dealt simultaneously with four branches of the energy spectrum, the two supplementary branches being the optical plasmon branch represented by the operator $\rho(P)$ and the acoustical plasmon branch represented by the operator $D(P)$. One can see that the equations of motion for the operators $d(P)$ and $d^{\dagger}(-P)$ reflect the interaction of excitons with optical and acoustical plasmons but do not contain the direct interaction between themselves. The interaction with acoustical plasmons also takes place through the quasiaverage constant $\tilde{\eta}$.

The equation of motion for the acoustical plasmon operator $D(P)$ contains the interaction with optical plasmons and the direct interaction with the magnetoexcitons. The optical plasmon motion is more separated from the equations of motion of other partners. It does not contain the direct interconnection with exciton branches, and the dispersion equation for the optical plasmon will be separated from the dispersion equation of the other three partners. In spite of this, optical plasmon branches are also influenced by the ground state of the system formed by the Bose-Einstein condensed magnetoexcitons with $k=0$, because the self-energy part $\Sigma_{33}(\vec{P},\omega)$ also depends on averages (23). On the contrary, the equation of motion of the acoustical plasmon operator is closely interconnected with the equation of motion of the both exciton operators and its energy spectrum cannot be found out separately. Unlike the usual theory of a Bose gas, the interaction between the magnetoexciton branches is not direct, but indirect, being mediated by the direct interaction with the plasmons. These peculiarities make our case different from those considered earlier in [58, 59, 67].

The cumbersome dispersion equation is expressed in general form by the determinant equation

$$\det\left|\Sigma_{ij}(\vec{P},\omega)\right|=0 \tag{26}$$

Due to the structure of the self-energy parts (22) it separates into two independent equations. One of them concerns only the optical plasmon branch and has a simple form

$$\Sigma_{33}(\vec{P},\omega)=0 \tag{27}$$

It does not include at all the chemical potential $\bar{\mu}$ and the quasiaverage constant $\tilde{\eta}$. The second equation contains the self-energy parts Σ_{11}, Σ_{22}, Σ_{44}, Σ_{14}, Σ_{41}, Σ_{24}, and Σ_{42}, which include the both parameters $\bar{\mu}$ and $\tilde{\eta}$. The second equation has the form

$$\Sigma_{11}(\vec{P};\omega)\Sigma_{22}(\vec{P};\omega)\Sigma_{44}(\vec{P};\omega)-\Sigma_{41}(\vec{P};\omega)\Sigma_{22}(\vec{P};\omega)\Sigma_{14}(\vec{P};\omega)-\Sigma_{42}(\vec{P};\omega)\Sigma_{11}(\vec{P};\omega)\Sigma_{24}(\vec{P};\omega)=0 \tag{28}$$

The solution of the equation (27) is

$$\left(\hbar\omega(P)\right)^2=\frac{\left\langle D(P)D(-P)\right\rangle}{N^2}\sum_Q U(Q)\left(U(-Q)-U(Q-P)\right)Sin^2\left(\frac{[P\times Q]_z l^2}{2}\right) \tag{29}$$

The right hand side of this expression at small values of P has a dependence $\left|P\right|^4$ and tends to saturate at large values of P. The optical plasmon branch $\hbar\omega_{OP}(P)$ has a quadratic dispersion law in the long wavelength limit and saturation dependence in the range of short wavelengths. Its concentration dependence is of the type $\sqrt{\mathrm{v}^2(1-\mathrm{v}^2)}$ what coincides with the concentration dependences for 3D plasma $\omega_p^2=\dfrac{4\pi e^2 n_e}{\varepsilon_0 m}$ [64] and for 2D plasma $\omega_p^2(q)=\dfrac{2\pi e^2 n_s q}{\varepsilon_0 m}$ [68], where n_e and n_s are the corresponding electron densities. The supplementary factor $(1-\mathrm{v}^2)$ in our case reflects the Pauli exclusion principle and the vanishing of the plasma oscillations at $\nu=\mathrm{v}^2=1$. The obtained dispersion law is represented in Fig. 2. A similar dispersion law was obtained in the case of 2D electron-hole liquid (EHL) in a strong perpendicular magnetic field [69], when the influence of the quantum vortices created by electron and hole subsystems is compensated exactly. But the saturation dependences in these two cases are completely different. In the case of Bose-Einstein condensed magnetoexcitons, it is determined by the ELLs, whereas in the case of EHL [69] it is determined by the Coulomb interaction in the context of the LLLs.

The solutions of the dispersion equation (28) describe the exciton energy branch, the exciton quasienergy branch and the acoustical plasmon branch. The ideal magnetoexciton gas can exist only in the case $\mathrm{v}^2=0$, with an infinitesimal number of excitons, but without plasma. The real parts $\sigma_{ij}(\vec{P},\omega)$ of the self-energy parts $\Sigma_{ij}(\vec{P},\omega)$ are

$$\begin{aligned}
&\sigma_{11}(\vec{P},\omega)=\hbar\omega+\bar{\mu}-E(P)+\Delta(P); && \bar{\mu}+\Delta(0)=0;\\
&\sigma_{22}(\vec{P},\omega)=\hbar\omega-\bar{\mu}+E(P)-\Delta(-P); && \tilde{\eta}=0;\\
&\sigma_{33}(\vec{P},\omega)=\sigma_{44}(\vec{P},\omega)=\hbar\omega; && \Delta(P)\approx\Delta(0);
\end{aligned} \tag{30}$$

The excitation of magnetoexcitons means to transfer one of them from the ground state with energy $-I_l$ to the excited state $-I_l+E(P)$. For this reason the magnetoexciton excitation spectrum equals $\hbar\omega_{ex}(P)=\pm E(P)$, whereas the plasma oscillation frequency vanishes $\hbar\omega=0$. This ideal variant is represented in Fig. 3. In the case of a non-ideal Bose-gas with $\nu\neq 0$, the self-energy parts contain terms linear in $U(P)$ and a term quadratic in the interaction constant with unknown frequency in the denominators under the summation symbols. These terms increase the number of the solutions, but can also be taken into account by an iteration method. In this case, one can obtain corrections to the earlier solutions.

The first step in this procedure gives the real parts of the self-energy parts

$$\begin{aligned}
&\sigma_{11}(\vec{P},\omega)=\hbar\omega+\bar{\mu}-E(P)+\Delta(P);\\
&\sigma_{22}(\vec{P},\omega)=\hbar\omega-\bar{\mu}+E(-P)-\Delta(-P);\\
&\sigma_{41}(\vec{P},\omega)=-\tilde{\eta}+U(P)\frac{\langle d(0)\rangle}{\sqrt{N}};\\
&\sigma_{42}(\vec{P},\omega)=\tilde{\eta}-U(-P)\frac{\langle d(0)\rangle}{\sqrt{N}};\\
&\sigma_{14}(\vec{P},\omega)=-2\tilde{\eta};\quad \sigma_{24}(\vec{P},\omega)=2\tilde{\eta};\\
&\sigma_{44}(\vec{P},\omega)=\hbar\omega;
\end{aligned} \tag{31}$$

The dispersion laws for two exciton branches and the acoustical plasmon branch are

$$\begin{aligned}
&\hbar\omega=\pm\sqrt{\left(\bar{\mu}-E(\vec{P})+\Delta(0)\right)^2+4\tilde{\eta}\left(\tilde{\eta}-\frac{U(\vec{P})\langle d(0)\rangle}{\sqrt{N}}\right)}\\
&\hbar\omega_{AP}(P)=0
\end{aligned} \tag{32}$$

In [37] the coefficient $(B_{i-i}-2A_{i-i}+\Delta(0))/I_l$ was determined to be $0.05r$. The rate r is $r=I_l/\hbar\omega$. The main parameters $(\bar{\mu}+\Delta(0))$, $\tilde{\eta}$ and $\frac{U(0)\langle d(0)\rangle}{\sqrt{N}I_l}=2\frac{A_{i-i}u\mathrm{v}}{I_l}$ are

$$(\bar{\mu}+\Delta(0))=0.1r\mathrm{v}^2I_l;\ \tilde{\eta}=-0.1r\mathrm{v}^3I_l;\qquad \frac{U(0)\langle d(0)\rangle}{\sqrt{N}}=0.3ru\mathrm{v}I_l;\qquad u^2+\mathrm{v}^2=1 \tag{33}$$

Expression (32) was found as follows:

$$\frac{\hbar\omega}{I_l}=\pm\sqrt{\left(0.1r\mathrm{v}^2-\frac{E(P)}{I_l}\right)^2+0.4r\mathrm{v}^3\left(0.1r\mathrm{v}^3+0.3ru\mathrm{v}e^{-\frac{P^2l^2}{2}}\right)} \tag{34}$$

In the limit $P\to 0$ there is a gap in the energy spectrum

$$\hbar\omega_{ex}(0)=2\sqrt{|\tilde{\eta}|U(0)\frac{\langle d(0)\rangle}{\sqrt{N}}}=0.346r\mathrm{v}^2\sqrt{u}I_l \tag{35}$$

It depends on the Hartree term of the overall attractive interaction in the system proportional to $-U(P)$, with $U(P)>0$, as well as on the quasiaverage theory parameter $\tilde{\eta}$ and on the amplitude of the condensate $\langle d(0)\rangle/\sqrt{N}$.

Unlike the case of a simple Bose gas with repulsive interactions, the collective excitations of the magnetoexcitons in a BEC with $k=0$ need a finite amount of energy. The magnetoexciton subsystem is incompressible when only the excitons themselves are taken into account and compressible when the optical plasmon branch is excited. In this approximation, the acoustical plasmon branch vanishes.

The energy spectrum described by expressions (32) and (34) begins with a gap in the limit $P\to 0$ and has a roton-type behavior with a minimum in the point P_l, determined by the equality $E(P_1)=0.1r\mathrm{v}^2 I_l$ and the minimal value $\hbar\omega_{ex}(P_1)=I_l\sqrt{0.12r^2u\mathrm{v}^4e^{-P_1l^2/2}}$. After the minimum the dispersion law transforms gradually in the energy spectrum of a free magnetoexciton. It is represented in Fig. 4 for a specific value of the rate $r=1/2$. This energy spectrum can be compared with another spectrum obtained by Girvin, MacDonald and Platzman [19] in the case of a two-dimensional one-component plasma (2DOCP) under the conditions of the FQHE. Both of them have the gaps, roton-type minima and saturation-type dependences at $P\to\infty$. We believe that the gaps are connected with the existence of the attractions in the systems.

In the case of the FQHE, the electrons have an attractive interaction with their correlation quasiholes formed by the many-fold vortices attached to each electron. As was mentioned by Lee and Zhang [25], if this interaction was neglected the energy spectrum could be gapless with quadratic dispersion law in the region $P\to 0$.

In the case of a coplanar 2D e-h system under the conditions of magnetoexciton formation, the attraction is created by the influence of the ELLs. The magnetoexcitons are embedded in the attractive bath with activation energy $U(0)=2A_{i-i}$. The exchange Fock terms and partially the Bogoliubov *u-v* transformation terms give rise to a repulsive interaction. Taking into account the quadratic terms of the self-energy parts (22) proportional to the constants $U(Q)U(P-Q)$, $U(Q)\tilde{W}(P-Q)$ and containing the unknown value $\hbar\omega$ in the denominators increases the number of solutions, which in our case is doubled. Two branches of the exciton-type collective elementary excitations are drawn in Fig. 5 by solid lines. Here the dotted line shows also the main exciton-type energy branch. The lower dispersion curve in Fig. 5 coincides in fact with the main exciton branch designed in Fig. 4. The second branch in Fig. 5 is practically parallel to the lower energy branch, being shifted on the energy scale by the value $U(0)$. We believe that this branch describes the more complicated process in which the previous excitation with wave vector P is accompanied by the excitation of the a second magnetoexciton with $k=0$ from the condensate in the bath, adding to it an activation energy $U(0)$ and transforming it into a free magnetoexciton with the same wave vector $k=0$, but outside of the attractive bath. As a result of this double excitation, two magnetoexcitons from the condensate with $k=0$ embedded in the attractive bath are transformed into an elementary excitation with wave vector $P\neq 0$ and another magnetoexciton with $k=0$ but outside the bath. The supplementary energy needed for this is the activation energy $U(0)=0.306rI_l$.

The acoustical plasmon branch has a dispersion law completely different from the optical plasmon oscillations. It has an absolute instability beginning with small values of wave vector going on up to the considerable value $pl\approx 2$. In this range of wave vectors, the optical plasmons have energies which do not exceed the activation energy $U(0)$. It means that the optical plasmons containing the opposite-phase oscillations of the electron and hole subsystems without displacement as a whole of their center of mass are allowed in the context of the attractive bath. On the other hand, the in-phase oscillations of the electron and hole subsystems in the composition of the acoustical plasmons are related to the displacements of their center of mass. Such displacements can take place only if their energy exceeds the activation energy $U(P)$. As a result, the acoustical plasmon branch has an imaginary part represented by the dashed line and is completely unstable in the region of wave vectors $pl \leq 2$. At greater values $pl > 2$ the energy spectrum is real and nonzero, approaching to the energy spectrum of the optical plasmons. The influence of the attractive bath $U(P)$ is represented by the dotted line. It vanishes in the limit $p \to \infty$. These properties of the acoustical plasmon branch are reflected in Fig. 6.

The starting Hamiltonian (10) has two continuous symmetries. One is the gauge global symmetry U(1) and another one is the rotational symmetry SO(2). The resultant symmetry is U(1)×SO(2). The gauge symmetry is generated by the operator N of the full particle number, when it commutes with the Hamiltonian. It means that the Hamiltonian is invariant under the unitary transformation $U(\varphi)$ as follows

$$U(\varphi)HU^{-1}(\varphi) = H; \quad U(\varphi) = e^{iN\varphi}; \quad \left[H, N\right] = 0 \tag{36}$$

The operator $\hat{N}$ is named as symmetry generator. The rotational symmetry SO(2) is generated by the rotation operator $\hat{C}_z(\varphi)$ which rotates the in-plane wave vectors $\vec{Q}$ on the arbitrary angle φ around z axis, which is perpendicular to the layer plane and is parallel to the external magnetic field. Coefficients $W_{\vec{Q}}$, $U(\vec{Q})$, $V(\vec{Q})$ in formulas (6) and (9) depend on the square wave vector $\vec{Q}$ which is invariant under the rotations $\hat{C}_z(\varphi)$. This fact determines the symmetry SO(2) of the Hamiltonian (10). The gauge symmetry of Hamiltonian (10) after the phase transition to the Bose-Einstein condensation (BEC) state is broken as it follows from expression (16). In the frame of the Bogoliubov theory of quasiaverages it contains a supplementary term proportional to η. The gauge symmetry is broken because this term does not commute with the operator N. More so, this term is not invariant under the rotations $C_z(\varphi)$, because the in-plane wave vector $\vec{k}$ of the BEC is transformed into another wave vector rotated by the angle φ in comparison with the initial position. The second continuous symmetry is also broken. In such a way the installation of the Bose-Einstein condensation state with arbitrary in-plane wave vector $\vec{k}$ leads to the spontaneous breaking of the both continuous symmetries. We will discuss the more general case of $\vec{k} \neq 0$ considering the case of $\vec{k} = 0$ as a limit $\vec{k} \to 0$ of the cases with small values $\vec{kl}$<<1. One can remember that the supplementary terms in Hamiltonian (10) describing the influence of the ELLs are actual in the range of small values $\vec{kl}$ <0.5. Above, we established that the number of the broken generators (BGs) denoted as N_{BG} equals to two ($N_{BG} = 2$).

The Goldstone theorem [70] states that the breaking of the continuous symmetry of the system leads to the appearance in the energy spectrum of the collective elementary excitations of the gapless branch equivalent to the massless particle in the relativistic physics. It occurs because the system with continuous symmetry has initially a set of the degenerate minimal values of the potential energy leading to a set of degenerate vacuum states. For example, the dependence of the potential energy on the order parameter may be Mexican hat like. The selection of one single vacuum state among the manifold of vacua and the fixing of the order parameter phase takes place due to quantum fluctuations and breaks spontaneously the continuous symmetry. The excitations over the selected vacuum transferring the system to the adjacent vacua and changing only the phase of the order parameter without change of its absolute value does not need any energy in the long wavelengths limit. Just these circumstances lead to the appearance of the gapless dispersion laws. These branches of the collective elementary excitations are named as the Nambu-Goldstone (NG) modes [70, 71]. They are of two types. One of them, of the first-type has a linear (or odd) dispersion law in the range of the small wave vectors, whereas the second-type has a quadratic (or even) dependence on the wave vector in the same region. The number of the NG modes in the system with many broken continuous symmetries was determined by the Nielsen and Chadha [72] theorem. It states that the number of the first-type NG modes N_I being accounted once and the number of the second type NG modes N_{II} being accounted twice equals or prevails the number of broken generators N_{BG}. It looks as follows $N_I + 2N_{II} \geq N_{BG}$. In our case the optical plasmon branch has the properties of the second-type NG modes. We have $N_I = 0$; $N_{II} = 1$ and $N_{BG} = 2$. It leads to the equality $2N_{II} = N_{BG}$. The Goldstone theorem guarantees that the NG modes do not acquire mass at any order of quantum corrections. Nevertheless, sometimes soft modes appear, which are massless in the zeroth order, but become massive due to quantum corrections. They were introduced by Weinberg [73], who has shown that these modes emerge if the symmetry of an effective potential of zeroth order is higher than that of the gauge symmetry. Following [74], these modes are currently named as the quasi-Nambu-Goldstone modes, in spite of the fact that their initial name proposed by Weinberg was pseudo-NG modes. Georgi and Pais [75] demonstrated that the quasi-NG modes also occur in the cases in which the symmetry of the ground state is higher than that of the Hamiltonian.

The authors of [74] underlined that the spinor BEC are ideal systems to study the physics of the quasi-Nambu-Goldstone(NG) modes, because these systems have a great experimental manipulability and well established microscopic Hamiltonian. In [74], it was shown that the quasi-NG modes appear in a spin-2 nematic phase. The ground state symmetry of the nematic phase at zeroth order approximation is broken by quantum corrections, thereby making the quasi-NG modes massive. The number n of the quasi-NG modes was determined by Georgi and Pais [75] in the form of a theorem. It was explaned and represented in [74] as follows:

$$n = \dim(\tilde{M}) - \dim(M) \tag{37}$$

where $\tilde{M}$ is the surface on which the effective potential assumes its minimal values to the zeroth order and $\dim(\tilde{M})$ is the dimension of this surface. The dimension $\dim(M)$ determines the number of the NG modes. This implies that M is a submanifold of $\tilde{M}$ and n is the dimension of the complementary space of M inside $\tilde{M}$ [74]. Returning to the case of 2D magnetoexcitons in the BEC state with wave vector $\vec{k}$ different from zero described by Hamiltonian (16), one can remember that both continuous symmetries existing in the initial form (10) are lost. It occurs because of the presence of the term $\tilde{\eta}(d^\dagger_{\vec{k}} + d_{\vec{k}})$ in the frame of the Bogoliubov theory of quasiaverages. Nevertheless, the energy of the ground state, as well as the self-energy parts $\Sigma_{ij}(P,\omega)$, which determine the energy spectrum of the collective elementary excitations, depends only on the modulus of the wave vector k and does not depend at all on its direction. All these expressions have a rotational symmetry SO(2), in spite of the fact that

Hamiltonian (16) has lost it. On our mind we have the condition described by Georgi and Pais [75] favoring the emergence of the quasi-NG modes. We are explaining the existence of the gapped, massive exciton-type branches of the collective elementary excitations obtained in our calculations just by these considerations.

We now discuss the damping rates of the obtained solutions and other possibilities related with the existence of the quantum vortices as a result of possible quantum fluctuations.
The damping rates of the obtained energy branches are determined by the imaginary parts of the self-energy parts

$$\Sigma_{ij}(\vec{P},\omega)=\sigma_{ij}(\vec{P},\omega)+i\Gamma_{ij}(\vec{P},\omega) \tag{38}$$

In the case of diagonal self-energy parts they are

$$\begin{aligned}&\Gamma_{11}(\vec{P},\omega)=\Gamma_{22}(-\vec{P},-\omega)=\\&=\frac{\langle D(P)D(-P)\rangle}{N^2}\pi\sum_Q U^2(Q)Cos^2\left(\frac{[P\times Q]_z l^2}{2}\right)\delta\left(\hbar\omega+\bar{\mu}-E(P-Q)+\Delta(P-Q)\right);\\&\Gamma_{33}(\vec{P},\omega)=\Gamma_{44}(\vec{P},\omega)=0\end{aligned} \tag{39}$$

The damping rates $\Gamma_{11}(\vec{P},\omega)$ and $\Gamma_{22}(\vec{P},\omega)$ are nonzero in complementary regions of the frequencies and wave vectors and can be calculated using the zero order dependences $\omega_{ex}(P)$ represented in Fig. 3 without taking into account the fine details revealed in Fig. 4.

The absolute values of the damping rates are drown in Fig. 7. They are smaller than the corresponding real parts represented in Fig. 4, which means that the obtained results have a physical sense. The damping rate of the optical plasmon branch in our description equals zero.
But there are also other possibilities in addition to equations of motion (22). They are related with the noncommutativity of the density operators $\rho(P)$ and $\rho(Q)$. This is given by

$$\rho(\vec{P}-\vec{Q})\rho(\vec{Q})=\rho(\vec{Q})\rho(\vec{P}-\vec{Q})+2iSin\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)\rho(\vec{P}) \tag{40}$$

and permits us to rewrite the nonlinear term into another form selecting a linear term as follows:

$$\begin{aligned}&-i\sum_Q W(Q)Sin\left(\frac{[P\times Q]_z l^2}{2}\right)\left[\rho(P-Q)\rho(Q)+\rho(Q)\rho(P-Q)\right]=\\&=\frac{1}{m}E(P)\rho(P)-i\sum_Q W(Q)Sin\left(\frac{[P\times Q]_z l^2}{2}\right)\times\\&\times\left[\left(1-\frac{1}{m}\right)\rho(P-Q)\rho(Q)+\left(1+\frac{1}{m}\right)\rho(Q)\rho(P-Q)\right];\\&E(P)=2\sum_Q W(Q)Sin^2\left(\frac{[P\times Q]_z l^2}{2}\right);\\&m=\pm1;\pm2;\pm3...\quad 0<|\nu|=\frac{1}{|m|}\leq 1\end{aligned} \tag{41}$$

Variant (22) corresponds to $m\to\infty$ and $\nu<1$. Adopting variant (41) we could use the zero-order Green's function with proper energy $E(p)/m$. We believe that this value is of the same type as the interaction energy of one electron with one vortex from the composition of the m-fold vortex attached to the electron. Although the influence of the vortices created by electrons and holes is compensated in the mean-field approximation, they can appear in the form of quantum fluctuations. Instead of the plasmon-type spectrum with the square frequency proportional to the electron-hole concentration we will obtain another type dependence reflecting the existence of vortices in the form of quantum fluctuations. The energy spectrum and the damping rates would be different from our results discussed above, but can be obtained exactly in the same way. Similar possibilities appear for the exciton-type operators. The assembly of new composite states arises.

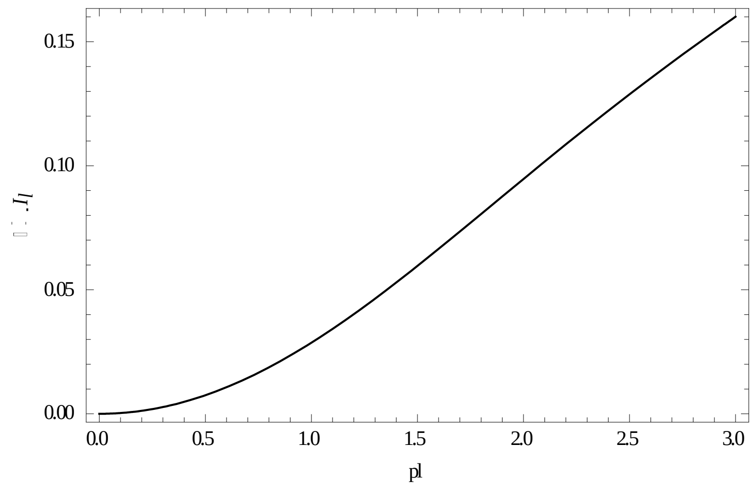

Fig. 2. The energy spectrum of the optical plasmon branch in the system of Bose-Einstein condensed magnetoexcitons with wave vector $\vec{k}=0$, filling factor $\nu = v^2 = 0.1$, under the influence of excited Landau levels with parameter $r = 1/2$.

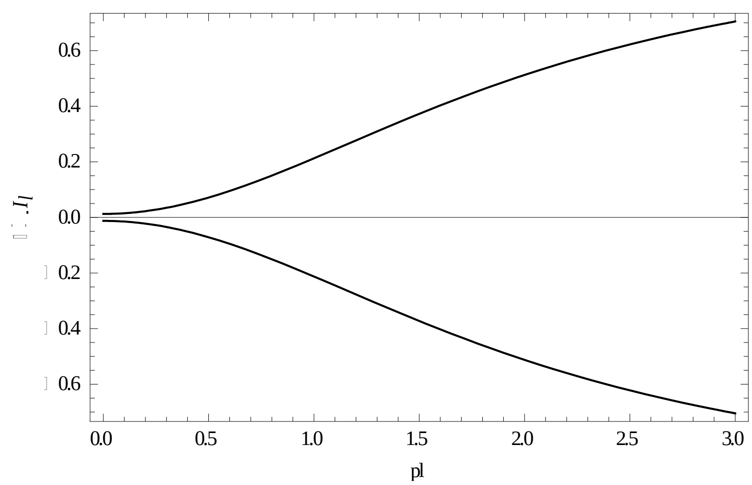

Fig. 3. The energy spectrum of the exciton branches in the case of an ideal BEC of magnetoexcitons with $\vec{k}=0$ and filling factor equal to zero.

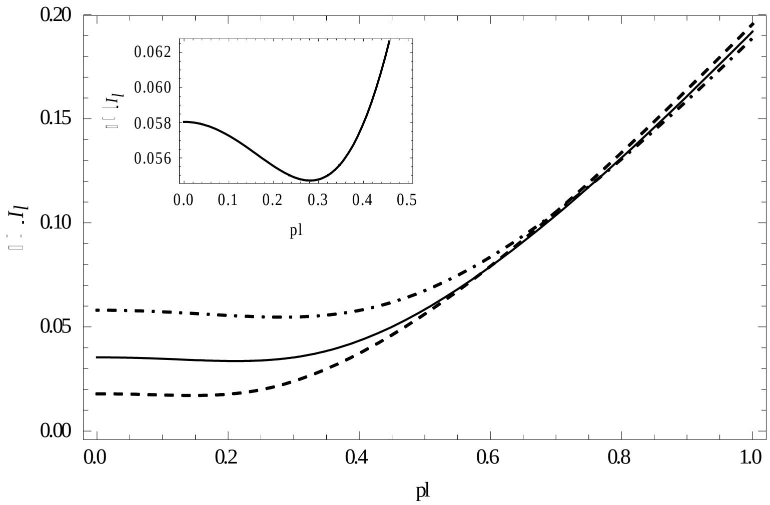

Fig. 4. The energy spectrum of the exciton branches in the case of a BEC of magnetoexcitons with $\vec{k}=0$ under influence of ELLs with parameter $r=1/2$, without activation processes. The solid line corresponds to the energy spectrum with filling factor $\nu=\mathrm{v}^2=1/5$, the dashed line corresponds to the filling factor $\nu=\mathrm{v}^2=1/10$, and the dot-dashed line to $\nu=\mathrm{v}^2=1/3$.

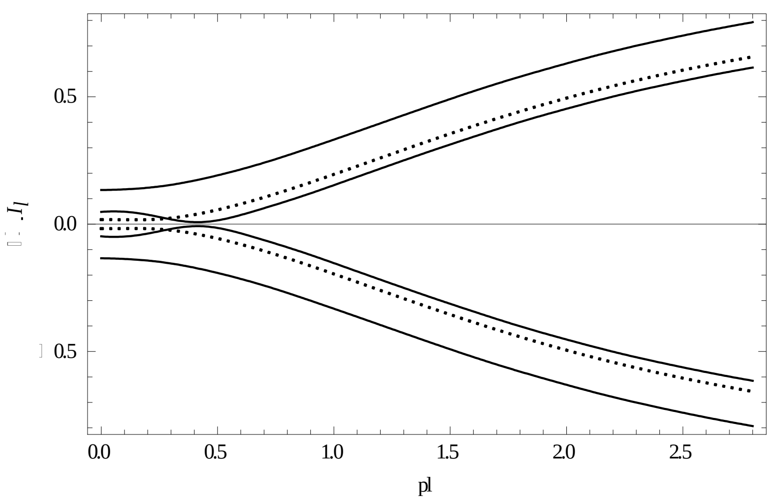

Fig. 5. The energy spectrum of the exciton branches in the case of a BEC of magnetoexcitons with $\vec{k}=0$ and filling factor $\nu=\mathrm{v}^2=0.1$ under the influence of ELLs with parameter $r=1/2$, taking into account the supplementary activation of the condensed magnetoexcitons and the formation of the second exciton branch. The dotted line represents the solution of Fig. 3.

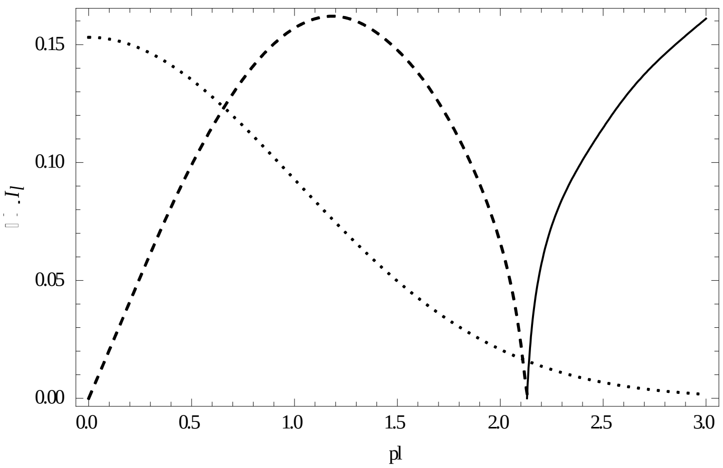

Fig. 6. The dispersion law of the acoustical plasmon branch. The solid line represents the real part, and the dashed line corresponds to the imaginary part. The dotted line gives the value $U(P)$.

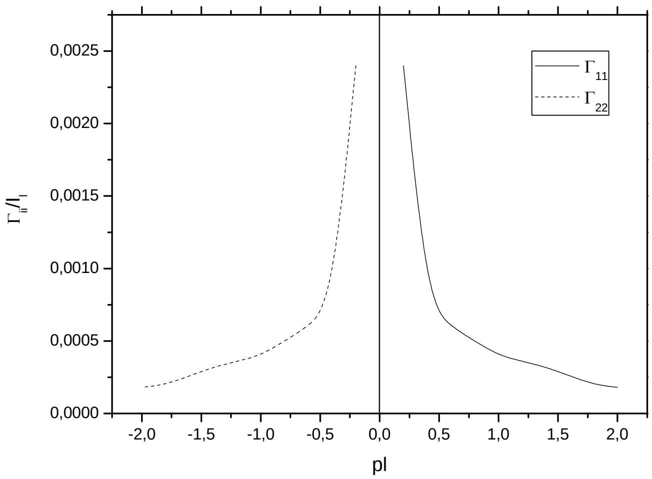

Fig. 7. The damping rates $\Gamma_{11}(\vec{P},\omega)$ and $\Gamma_{22}(\vec{P},\omega)$ of the diagonal self-energy parts.

6. Conclusions

The energy spectrum of the collective elementary excitations of a 2D electron-hole (e-h) system in a strong perpendicular magnetic field in a state of Bose-Einstein condensation (BEC) with wave vector $\vec{k}=0$ has been investigated in the framework of Bogoliubov theory of quasiaverages. The starting Hamiltonian describing the e-h system contains not only the Coulomb interaction between the particles lying on the lowest Landau levels, but also a supplementary interaction due to their virtual quantum transitions from the LLLs to the excited Landau levels and back. This supplementary interaction generates, after the averaging on the ground state BCS-type wave function, direct Hartree-type terms with an attractive character, exchange Fock-type terms giving rise to repulsion, and similar terms arising after the Bogoliubov u-v transformation. The interplay of these three parameters gives rise to the resulting nonzero interaction between the magnetoexcitons with wave vector $\vec{k} = 0$ and to stability of their BEC.

The equations of motion for the exciton operators $d(P)$ and $d^{\dagger}(P)$ are interconnected with equations of motion for the density operators $\rho(P)$ and $D(P)$. Instead of a set of two equations of motion as in the case of usual Bose gas, corresponding to normal and abnormal Green's functions, we have a set of four equations of motion. The change of the center-of-mass wave vector of the magnetoexciton, for example from 0 to $\vec{P}$, means the change of its internal structure because the internal distance between the Landau orbits of the quantized electron and hole becomes equal to $|\vec{P}| l^2$.

The separated electrons and holes remaining on their Landau orbits can take part in the formation of magnetoexcitons as well as in collective plasma oscillations. These possibilities were not taken into consideration in the theory of structureless bosons or in the case of Wannier-Mott excitons with a rigid relative electron-hole motion structure without the possibility of the intra-series excitations. Magnetoexcitons have an internal structure that is much less rigid than standard Wannier-Mott excitons and the possibilities for electrons and holes to take part simultaneously in many processes are much more diverse. Instead of the branches of the energy spectrum corresponding to normal and abnormal Green's functions, we have to deal simultaneously with four branches of the energy spectrum, the two supplementary branches being the optical plasmon branch represented by the operator $\rho(P)$ and the acoustical plasmon branch represented by the operator $D(P)$.

The energy spectrum of the collective elementary excitations consists of four branches. Two of them are excitonic-type branches, one of them being the usual energy branch whereas the second one is the quasienergy branch representing the mirror reflection of the energy branch. The other two branches are the optical and acoustical plasmon branches. The exciton energy branch has an energy gap due to the attractive interaction terms, which needs to be overcome for excitation, as well as a roton-type region in the range of intermediary values of the wave vectors. At higher values of wave vector, its dispersion law tends to saturation. The optical plasmon dispersion law is gapless with quadratic dependence in the range of small wave vectors and with saturation-type dependence in the remaining part of the spectrum. The acoustical plasmon branch reveals an absolute instability of the spectrum in the range of small and intermediary values of the wave vectors. In the remaining range of the wave vectors, the acoustical plasmon branch has a real value of the energy spectrum approaching the energy spectrum of the optical plasmon branch in the limiting case of great wave vectors.

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